GROUP ALGEBRAIC STUDY OF PERMUTATION - INVARIANT FILTERS

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by

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CERTIFICATE

Certified that this work 'GROUP ALGEBRAIC STUDY OF PERMUTATION–INVARIANT FILTERS' by Mr. R. Hari Prasada Rao has been carried out under my supervision and that this has not been submitted elsewhere for a degree.

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Professor

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ABSTRACT

Group algebraic approach to the study of P–I filters is considered. Classification of P–I filters based on the ideals in group algebra is proposed and the possibility of equivalent realization of cyclic permutation invariant (P–I) filters in terms of abelian P–I filters is investigated. It is shown that such realizations are possible only for those P–I filters which are invariant relative to both cyclic and abelian permutation groups of same degree. Equivalent realizations are obtained on the basis of Mixed–Radix (MRX) mapping and mapping based on Chinese–Remainder Theorem (CRT) for integers. Complete characterization of such filters is given for both of these mappings. Implementational advantages of realizing cyclic P–I filters as abelian P–I filters from the point of view of structural concurrency are pointed out.

In the group algebra representation of a signal space, the space consisting of all n–length real–valued sequences, a P–I filter is a mapping from group algebra into an ideal of the group algebra. A class of P–I filters that maps elements from a group algebra into an ideal is itself shown to be an ideal. As a consequence, cyclic P–I filters are classified based on the ideals in cyclic group algebra and abelian P–I filters are classified based on the ideals in abelian group algebra. An expression for the number of conjugacy classes in abelian group algebra is given and its significance to the above classification is pointed out. This classification is instrumental in the identification of filters that are invariant under both cyclic and abelian permutations. Such filters are interpreted as elements of cyclically closed ideals in abelian group algebras and are called *cyclically closed abelian P–I filters*. MRX mapping provides a basis to the characterization of such ideals. From these ideals, a complete characterization of cyclically closed abelian P–I filters is obtained.

Whenever the orders of component cyclic sub groups are pairwise relatively prime integers, mapping based on CRT for integers is considered and it is shown that any cyclic P–I filter, whose dimension is a product of pairwise relatively prime integers has, an equivalent abelian P–I filter realization. It is shown that the same sample domain CRT mapping relates the respective transform domain coefficients. Therefore, in the transform domain, this equivalent abelian P–I filter realization has an implementation advantage over traditional cyclic P–I filter realization using Good–Thomas FFT algorithm in respect of address shuffling.

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To

My Parents and Teachers

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LIST OF SYMBOLS

Z_n : Set of integers modulo n

$\mathbb{R}$: Field of real numbers

$\mathbb{R}^n$: n–dimensional vector space over $\mathbb{R}$

a : an n–tuple belonging to $\mathbb{R}^n$

M : Mixed–Radix set $\{ m_0 , m_1 , .., m_{r-1} \}$ of n

$< i_0 \, i_1 \ldots i_{r-1} >$: Mixed–Radix representation of i w.r.t. M

$\ominus$: Subtraction in Mixed–Radix number system

PG : Transitive abelian permutation group of degree n

PC : Transitive cyclic permutation group of degree n

C_{m_i} : Cyclic group of order m_i

x_i : Generator of C_{m_i}

$\otimes$: Direct product

G : Abelian group, $(= C_{m_0} \otimes C_{m_1} \otimes \ldots \otimes C_{m_{r-1}})$, of order n

X : General element of G

X^i : $x_0^{i_0} . x_1^{i_1} \ldots x_{r-1}^{i_{r-1}}$

RG : Group algebra of G over $\mathbb{R}$

a(X) : General element of RG

Z[a(X)] : Cyclic shift of a(X)

X^i[a(X)] : Abelian permutation of a(X)

$\theta(x_i)$: Minimal idempotent of C_{m_i} algebra.

$R_0(x_i)$: Formal sum consisting of all the elements in C_{m_i}

$\theta(X)$: Idempotent generator of RG

$<\theta(x_i)>$: Ideal generated by $\theta(x_i)$

C_n : Cyclic group of order n

Z : Generator of C_n

RC : Cyclic group algebra over $\mathbb{R}$

a(Z) : General element of RC

$\oplus$: Direct sum

$\mathbb{C}$: Complex space

$\mathbb{C}_r^n$: Complex space isomorphic to $\mathbb{R}^n$

CG_r : Group algebra over $\mathbb{C}_r^n$

t_i : Number of conjugacy classes in C_{m_i} algebra

$E^j(X_i)$: Minimal idempotent generator of C_{m_i} algebra over a complex space restricted to the real images

$E^j(X)$: J^{th} minimal idempotent of CG_r

$<E^j(X_i)>$: Ideal generated by $E^j(X_i)$

CC_r : Cyclic group algebra over $\mathbb{C}_r^n$

$E^j(Z)$: Minimal idempotent generator of CC_r

$<E^j(Z)>$: Ideal generated by $E^j(Z)$

$\square$ End of proof

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CHAPTER 1
INTRODUCTION

1.1 Motivation

Signals are defined as real–valued functions of a variable that takes values from an infinite index set. This index set has the structure of an infinite abelian group. A linear time–invariant (LTI) system is a linear convolution operator that convolves an input signal with its impulse response signal thereby producing an output signal. Fourier transform (FT) provides a basis for the transform domain study of LTI systems. Filtering of a signal by an LTI system means modification of the fourier spectrum of the input signal by an appropriately chosen LTI system. To process finite length signals using LTI systems, zeros are appended to the signals and these are treated as signals defined over infinite index set with a finite support. Alternatively index set can be treated as finite to deal with finite length signals. When the structure of index set is a finite cyclic group, we have a known class of cyclic convolution systems (CCSs). Here, discrete fourier transform (DFT) plays a role similar to that of FT for the case of LTI systems. The basis functions for DFT are discrete complex exponentials and are ordered in terms of frequency of discrete complex exponentials. Hence the notion of filtering with respect to frequency as low pass, high pass, band pass and band stop is easily introduced. Another well–known class of dyadic convolution systems (DCSs) are obtained by assuming finite dyadic group structure to the finite index set. Walsh–Hadamard transform (WHT) provides the necessary basis for the transform domain study of this class of systems. Here also the discrete Walsh functions which are the basis functions for WHT are ordered in terms of number of zero crossings generally known as sequency. Therefore, analogous to the notion of filtering with respect to frequency using CCSs, we have the notion of filtering using DCSs with respect to sequency.

Generalization of the structure of finite index set to a finite abelian group results in a more general class of convolution systems known as permutation invariant (P–I) convolution systems. In such systems, generalized Walsh–Hadamard transforms (GWHTs) play a role similar to that of DFT and WHT respectively in the context of CCSs and DCSs. In fact, CCSs and DCSs are special cases of the general class of P–I convolution systems. Filtering of signals using P–I convolution system that does not belong to either cyclic class or dyadic class, means modifying the GWHT spectrum of input signals by an appropriate P–I convolution system. In this general case, the criterion of number of zero crossings may be used to find an ordering of the basis functions for GWHT as does the notions of frequency and sequency in the case of CCS and DCS P–I systems respectively. It remains to be seen whether this provides a viable alternative. It is obvious that for such GWHT, the number of zero crossings of basis functions will be lower bounded by the zero crossings (frequency) of a corresponding DFT basis functions and upper bounded by the zero crossings (sequency) of the corresponding basis functions of WHT (whenever it is applicable). In this context, it is worth while to compare the effects of filtering by different classes of P–I filters. Towards this end, experimentally, given an image, some of the conjugate positions were forced to zero, and the effect of this type of filtering was compared both with respect to cyclic and non–cyclic class of P–I systems of the same order. It was noticed that upon forcing certain conjugacy classes to zero produced same output images in both cases. This observation motivated to study whether there are any systems that belong to both cyclic and abelian classes and the possibility of equivalent realization of cyclic permutation invariant (P–I) systems in terms of abelian P–I systems.

1.2 Scope of the Work

This thesis is concerned with the study of equivalent realization of cyclic P–I filters in terms of abelian P–I filters. For a given class of abelian P–I filters, we obtained a complete characterization of P–I filters that are invariant under both cyclic and abelian

permutations. For this characterization, MRX mapping provides the necessary basis. However, when the orders of the component cyclic subgroups are pairwise relatively prime integers, mapping based on CRT is also possible and it is shown that all cyclic P–I filters whose dimension are products of integers which are pairwise relatively prime have an equivalent abelian P–I filter realizations under this mapping.

Signal space, the space consisting of all n–length real valued sequences, can be viewed either as a vector space, $\mathbb{R}^n$, or as a group algebra, RG, where G is an abelian group of order n. A finite discrete system is a linear transformation, T, which is a mapping from $\mathbb{R}^n$ into $\mathbb{R}^n$. A system is said to be permutation invariant (P–I), if the input signal is permuted in a particular manner then, the output signal of the system is also permuted in the same manner. On given number of n–samples of an input signal, there exists n! permutations. Each class of P–I systems of dimension n is defined with respect to a set of permutations, forming transitive abelian permutation (TAP) group, PG, of degree n. The number of distinct classes of P–I systems of dimension n is equal to the number of distinct TAP groups of degree n, in the n! permutations. A P–I system, T, defined invariant relative to a TAP group, PG, of degree n which is isomorphic to G, is called an abelian P–I filter if it maps any n–tuple of $\mathbb{R}^n$ into an n–tuple of a linear subspace of $\mathbb{R}^n$ which is closed under the permutations belonging to PG. Input signal space to an abelian P–I filter is the whole $\mathbb{R}^n$ and output signal space is a linear subspace of $\mathbb{R}^n$ closed under permutations of PG. A cyclic P–I filter is a P–I system, T, defined invariant relative to a cyclic TAP group, PC, of degree n, whose output signal space is a linear subspace of $\mathbb{R}^n$ closed under cyclic permutations belonging to PC. Linear P–I filters have matrix representations and these matrices are called P–I matrices. The zeroth column of a P–I matrix is the unit–sample response of the filter and other columns of this matrix are just the permutations (of the TAP group w.r.t. which the filter is defined) of the zeroth column. Any output signal of a P–I filter is a linear combination of the column vectors of the P–I matrix with the samples of the input signal as weighting factors

Suppose, if on a particular vector of $\mathbb{R}^n$, the effect of an abelian permutation of PG is same as that of a cyclic permutation of PC, then a P–I filter whose P–I matrix has got this type of vector as its zeroth column maps any n–tuple of $\mathbb{R}^n$ into an n–tuple of a linear subspace of $\mathbb{R}^n$ closed under both abelian as well as cyclic permutations belonging to PG and PC respectively. Such a P–I filter is called a cyclically closed abelian P–I filter. The main objective of this thesis is the characterization of such P–I filters based on both MRX and CRT mapping. For characterization purposes it will be advantageous to work in the group algebraic framework.

1.3 Proposed Line of Approach

There are three basic steps involved towards characterizing cyclically closed abelian P–I filters. The first step is to decompose $\mathbb{R}^n$ into a direct sum of smallest subspaces closed under the permutations belonging to PG. The second step is to identify which of these smallest subspaces or direct sum of subsets of these smallest subspaces, are closed under cyclic permutations belonging to PC. The last step is to identify the vectors whose cyclic permutations are same as abelian permutations from these spaces. All these steps are carried out by making use of the properties of group algebraic structure of signal space. Then P–I filtering is nothing but multiplication of group algebraic representations of input and filter vectors. Group algebra, RG, of a group G which is isomorphic to PG, over the field of real numbers can be expressed as a direct sum of minimal ideals in that group algebra. Any other ideal is a direct sum of subset of these minimal ideals. Every minimal ideal has an unique idempotent generator. This generates all the elements of the corresponding minimal ideal. Characterization of these idempotent generators is done easily in the transform domain. GWHT is an isomorphism between RG and the complex space restricted to the real images. In GWHT domain each of the minimal ideals in RG corresponds to a set of elements which are non–zero over a single common conjugacy class. By invoking some properties of minimal ideals, it is shown that minimal ideals are

isomorphic to smallest subspaces closed under the permutations belonging to PG and that there is one to one correspondence between the ideals in group algebra and the subspaces of $\mathbb{R}^n$ closed under the permutations belonging to PG. This fact leads to the interpretation that abelian P–I filter is a mapping from RG to an ideal and a set of filters whose transform vectors are non–zero over a single common conjugacy class forms a minimal ideal. Consequently, cyclic P–I filters are classified based on the ideals in cyclic group algebra and abelian P–I filters are classified based on the ideals in abelian group algebra. This classification is made use of in the characterization of cyclically closed abelian P–I filters. For a given class of abelian P–I filters, we characterize all the elements of abelian group algebra whose cyclic permutations are same as the corresponding abelian permutations. This correspondence is established on the basis of mixed–radix (MRX) mapping. Because of the necessity that these elements should belong to the ideals of abelian group algebra that are closed under cyclic permutations, firstly, we completely characterize all the ideals that are closed under cyclic permutations in the abelian group algebra. This characterization is given in [1] for the group algebra over a finite field. It is noticed that this result is also applicable to the case of Real field and is used for characterizing cyclically closed abelian ideals in RG. Since, not all the elements of cyclically closed ideals in abelian group algebra qualify as cyclically closed abelian P–I filters, we characterize all those elements which are cyclically closed abelian P–I filters. Further, we consider CRT mapping (When the orders of component cyclic groups are pair wise relatively prime) and show that under this mapping, a cyclic group algebra is isomorphic to an abelian group algebra. Then we show that all cyclic P–I filters in such a cyclic group algebra have an equivalent realization as an abelian P–I filters. A brief summary of results obtained is given below.

1) P–I filters are classified based on the ideals in group algebra.

2) A complete characterization of cyclically closed abelian P–I filters is given based on MRX mapping.

3) A procedure for identification of 2–D cyclically closed abelian P–I filters is given.

4) It is shown that any cyclic P–I filter whose dimension is a product of pairwise relatively prime integers can be realized as an equivalent abelian P–I filter.

1.4 Outline of Chapters

In Chapter 2, P–I filters are classified based on the ideals in group algebra. This chapter starts with the basic definitions of P–I filters. In Section 2.2, some of the important properties of group algebra are discussed and it is shown that a P–I filter's output signal space is an ideal in a group algebra. In the next section, the set of all P–I filters that maps elements from RG into an ideal, is shown to be that ideal itself. This provides a basis for classifying P–I filters. Consequently, cyclic and abelian P–I filters are classified based on ideals in their respective group algebras. In Section 2.4, transform–domain characterization of ideals in a group algebra is reviewed. An expression giving the number of conjugacy classes is derived and its significance with respect to above mentioned classification of P–I filters is pointed out. In the last section of this chapter, transform–domain classification of P–I filters is discussed.

Chapter 3 is devoted to the characterization of cyclically closed abelian P–I filters. In Section 3.1, the concept of cyclically closed abelian P–I filter is explained. In the next section cyclically closed P–I filters are shown to be elements of cyclically closed ideals in abelian group algebra. Section 3.3, contains the characterization of cyclic ideals in an abelian group algebra. In the next section, the characterization of cyclically closed abelian P–I filters is given. In Section 3.5, transform–domain characterization of cyclically closed abelian P–I filters is given. In Section 3.6, 2–D P–I filters are discussed and a procedure for identifying 2–D cyclically closed abelian P–I filters is given. In the last section, we discussed equivalent abelian P–I filter realizations of cyclic P–I filters based on CRT mapping.

The thesis is concluded in Chapter 4 giving summary of results obtained and suggestions for further research.

CHAPTER 2
CLASSIFICATION OF P–I FILTERS BASED ON IDEALS IN GROUP ALGEBRA

A signal space consisting of all real–valued sequences of length n, can be treated either as a vector space over reals, $\mathbb{R}^n$, or as a group algebra over the real field, RG [4]. Usually, a signal space is treated as $\mathbb{R}^n$ so that a Permutation–invariant (P–I) system turns out to be a linear transformation on $\mathbb{R}^n$, T, which is a mapping of $\mathbb{R}^n$ into a P–I subspace of $\mathbb{R}^n$ [3]. On the other hand, when a signal space is treated as a group algebra, RG, a P–I system can be viewed as multiplication of elements belonging to the group algebra RG, with the unit sample response element $\in$ RG, characterizing the P–I system.

In this chapter, we consider group algebra representation of a signal space and consequently, classify P–I filters based on the ideals in group algebra.

In Section 2.1, we provide certain preliminary definitions that are required for our study. In Section 2.2, we consider group–algebra representation of a signal space and list out some of the important properties of group–algebra that are relevant for our study. In addition, we show that there is an isomorphic correspondence between the ideals of a group algebra and the P–I subspaces of the vector space $\mathbb{R}^n$. In Section 2.3, a class of P–I filters that maps elements of a group algebra into an ideal is shown to be that ideal itself. In Section 2.4, transform–domain characterization of ideals in RG is reviewed and an expression for the number of conjugacy classes is derived. Finally, in Section 2.5, we present the transform–domain classification of P–I filters.

2.1 Permutation–Invariant (P–I) Filters

In this section treating the signal space, the space consisting of all n–length real–valued sequences, as an n–dimensional vector space over the field of real numbers, $\mathbb{R}^n$, the concept of P–I filter is being explained.

Let us consider the following three n–tuples, $a = (a_0\ a_1\ a_2\ ..\ a_j\ ..\ a_{n-1})$, $b = (b_0\ b_1\ b_2\ ..\ b_i ..\ b_{n-1})$ and $s = (s_0\ s_1\ s_2\ ..\ s_j ..\ s_{n-1})$ from $\mathbb{R}^n$ denoting the input output and unit–sample response sequences of a system respectively.

Let PG be the transitive abelian permutation group of degree n. The effect of any permutation, p_k , belonging to PG on any signal, $a = (a_0\ a_1\ a_2\ ..\ a_j\ ..\ a_{n-1}) \in \mathbb{R}^n$, is described by the following relationship.

$$p_k(a) = < a_{0\theta k}\ a_{1\theta k}\ ..\ a_{j\theta k}\ ..\ a_{n-1\theta k} > \ ;\ k \in Z_n \qquad (2.1.1)$$

where $j\Theta k = < (j_0 - k_0)_{m_0}\ \ (j_1 - k_1)_{m_1} ..\ (j_{r-1} - k_{r-1})_{m_{r-1}} >$

$$(2.1.2)$$

A system, T, which is defined with respect to the transitive abelian permutation group, PG, of degree n, whose elements, p_k , $k \in Z_n$, are being specified by the mixed–radix set $\{m_0\ m_1\ ...\ m_{r-1}\}$, is characterized by the following generalized convolutional relationship.

$$b_i = \sum_{j=0}^{n-1} s_{i\theta j}\, a_j \qquad (2.1.3)$$

where $i\Theta j = < (i_0 - j_0)_{m_0}\ \ (i_1 - j_1)_{m_1} ..\ (i_{r-1} - j_{r-1})_{m_{r-1}} >$

Alternatively (2.1.3) can also be expressed in the form of matrix–equation as

b=Sa ; a and b are nx1 matrices and S is nxn matrix (2.1.4)

Elements of S are given by

$$S_{i,j} = s_{i\theta j} \qquad (2.1.5)$$

where $i\Theta j$ is as defined above.

Example 2.1.1: For the class of P–I systems of dimension n defined with respect to the cyclic permutation group PC of degree n, the generalized convolutional relationship takes the familiar form of the cyclic convolutional relationship

$$b_i = \sum_{j=0}^{n-1} s_{(i-j) \bmod n}\, a_j \quad ; \; i \in Z_n \tag{2.1.6}$$

representing cyclic P–I systems commonly known as cyclic convolution systems.

Hereafter, we refer to the above defined system as system, T, invariant relative to PG. It is understood that PG is being defined with respect to the mixed–radix set $\{m_0\ m_1\ \ldots\ m_{r-1}\}$.

The matrix, S, in (2.1.5) is also known as a permutation–invariant matrix, and its generator, i.e., the zeroth column, $s = (s_0\ s_1\ s_2\ ..\ s_j\ ..\ s_{n-1})$ is actually the unit–sample response of the system, T. Any other column, k, of S can be obtained by permuting the zeroth column, s, with the kth member, p_k, of PG, in accordance with (2.1.1). This fact leads to the interpretation that, the output sequence, b, of T can be interpreted as a linear combination of its column vectors with the samples of the input sequence, a, as weighting factors. In the form of equation, it is represented as

$$\begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_i \\ \vdots \\ b_{n-1} \end{bmatrix} = \sum_{k=0}^{n-1} a_k \begin{bmatrix} s_{0_\theta k} \\ s_{1_\theta k} \\ \vdots \\ s_{j_\theta k} \\ \vdots \\ s_{n-1_\theta k} \end{bmatrix} \tag{2.1.7}$$

In other words, the system, T, defined with respect to PG, maps the space of real n–tuples, $\mathbb{R}^n$, into the space of real n–tuples, in which each of the n–tuples is a linear combination of the column vectors of S. It means, the output signal space of the system, T,

has the following sequences:

a) Zeroth column, s, of S

b) All the resulting sequences, by the application of permutations belonging to PG to s.

c) The sequences obtained by the linear combinations of the sequences mentioned in (a) and (b).

This output signal space of T being characterized by s is linear and is closed under the permutations belonging to PG. In the following discussion, this type of space is simply referred to as the linear space invariant relative to PG. When the whole $\mathbb{R}^n$ is considered, it is obviously a linear space invariant relative to PG and therefore a valid output signal space for T. From the space of real n–tuples, $\mathbb{R}^n$, it is possible to find linear subspaces of $\mathbb{R}^n$ invariant relative to PG. These subspaces also can be taken as valid output signal spaces for the system, T, as defined in (2.1.3). In this context, the system, T, which is a mapping from $\mathbb{R}^n$ into a linear subspace of $\mathbb{R}^n$, invariant relative to PG, is called a P–I filter. In other words, P–I filter maps any real n–tuple belonging to $\mathbb{R}^n$ into a n–tuple belonging to the linear subspace of $\mathbb{R}^n$ invariant relative to PG. Summarizing the above discussion we have the following definition for P–I filter.

Definition 2.1.1: A *P–I filter* can be defined as a system, T, which maps the space of real n–tuples, $\mathbb{R}^n$, into the linear subspace of $\mathbb{R}^n$, invariant relative to PG and its output–input relationship is characterized by (2.1.3).

This P–I filter is said to be a *minimal P–I filter* if its output signal space does not contain any proper subspaces invariant relative to PG. Figure 2.1.1 illustrates the above notions of P–I system, P–I filter and minimal P–I filter.

If the permutation group PG is cyclic (denote it by PC), then a P–I filter is called a *cyclic P–I filter*. Otherwise, it is called as an *abelian P–I filter*. It is to be noted that the output signal space of a cyclic P–I filter is a linear subspace of $\mathbb{R}^n$ closed under the permutations of PC. On the other hand, output signal space of an abelian P–I filter is a

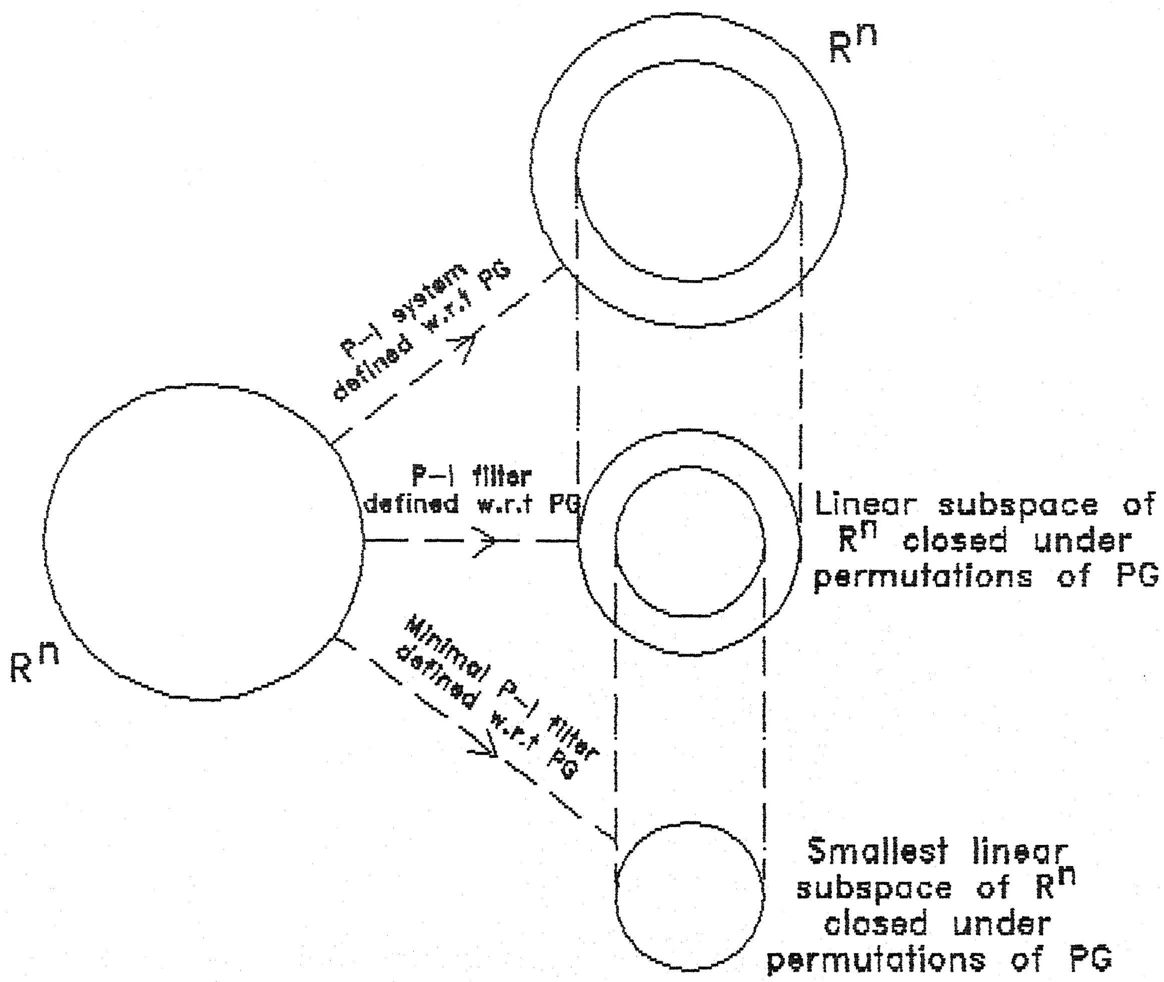

Fig 2.1.1 Illustration of the notions of P-I System, P-I Filter and Minimal P-I Filter.

2.2 Review of Group Algebraic Concepts

In this section the signal space is being treated as a group–algebra [4] and then isomorphic correspondence is established between the ideals in group algebra and the permutation invariant subspaces in $\mathbb{R}^n$. This section starts with indexing the elements of an abelian group using mixed–radix indexing scheme.

The elements of an abelian group, G, of order n which is a direct product of cyclic subgroups with generators $x_0, x_1, .., x_{r-1}$ of orders $m_0, m_1, .., m_{r-1}$ respectively, are indexed as follows.

$$G = C_{m_0} \otimes C_{m_1} \otimes ... \otimes C_{m_{r-1}} \quad .. \; ; n = m_0\, m_1 ... m_{r-1}$$

where C_{m_i} is cyclic subgroup of order m_i with generator x_i and $\otimes$ denotes direct product.

$$X^0 = 1$$
$$X^1 = x_0$$
$$X^2 = x_0^2$$
$$.$$
$$X^{m_0-1} = x_0^{m_0-1}$$
$$X^{m_0} = x_1$$
$$X^{m_0+1} = x_1 x_0$$
$$.$$

$$X^i = x_{r-1}^{i_{r-1}}\, x_{r-2}^{i_{r-2}} \dots x_1^{i_1}\, x_0^{i_0} = \prod_{\alpha=0}^{r-1} x_\alpha^{i_\alpha} \tag{2.2.1}$$

where i_α is the index of i with respect to the mixed–radix m_α in the mixed–radix representation of i with respect to the mixed–radix set $\{m_0, m_1, ..., m_{r-1}\}$

$$..$$

$$X^{n-1} = x_{r-1}^{m_{r-1}-1}\, x_{r-2}^{m_{r-2}-1} \dots x_1^{m_1-1}\, x_0^{m_0-1}$$

The i^{th} element X^i of the abelian group, G, is indexed using mixed–radix indexing

scheme as given in (2.2.1). In the following discussion X is used to denote a general element belonging to G.

Any n–tuple, $a = (a_0\ a_1\ a_2\ ..\ a_j\ ..\ a_{n-1})$ belonging to $\mathbb{R}^n$ can be expressed as a polynomial in variables $x_0, x_1, .., x_{r-1}$ as follows.

$$a(x_0, x_1, .., x_{r-1}) = \sum_{i=0}^{n-1} a_i \left(\prod_{\alpha=0}^{r-1} x_\alpha^{\ i_\alpha} \right) \tag{2.2.2}$$

where i_α is as defined in (2.2.1)

Let us denote the term in brackets of (2.2.2) by $X^i = \prod_{\alpha=0}^{r-1} x_\alpha^{\ i_\alpha}$. In the light of above discussion on indexing the elements of abelian group, G, of order n it is to be noted that the set of X^i s, $\{X^0, X^1, .., X^{n-1}\}$, has the structure of an abelian group of order n.

The polynomial $a(x_0, x_1, .., x_{r-1})$ can be interpreted as formal sums of the form $\sum_{i=0}^{n-1} a_i \left(\prod_{\alpha=0}^{r-1} x_\alpha^{\ i_\alpha} \right)$ where $a_i \in R$ and $x_\alpha^{\ i_\alpha} \in C_{m_\alpha}$, a cyclic group of order m_α with generator x_α. This set of formal sums have an algebraic structure called group algebra.

Definition 2.2.1: Let G be an abelian group of order n and $\mathbb{R}$ real field. The set of formal sums denoted by RG where $RG = \{a(X) = \sum_{X \in G} a_X X\ ;\ a_X \in \mathbb{R}\}$, with the two operations addition and multiplication respectively as

$$a(X) + s(X) = \sum_{X \in G} (a_X + s_X) X\ ;\ a(X) = \sum_{X \in G} a_X X\ ,\ s(X) = \sum_{X \in G} s_X X \tag{2.2.3}$$

and for $Y \in G$

$$Ya(X) = \sum_{X \in G} a_X XY = \sum_{X \in G} a_{XY^{-1}} X \tag{2.2.4}$$

is called group algebra.

Remark 2.2.1: Combining the operations given by (2.2.3) and (2.2.4) we get, for $A, B \in RG$

$$a(X)s(X) = \left(\sum_{X \in G} a_X X \right) \left(\sum_{X \in G} s_X X \right) = \sum_{W \in G} \left(\sum_{XY=W} a_X s_Y \right) W \tag{2.2.5}$$

The term inside the brackets of last expression of (2.2.5) is a convolution

determined by the structure of the abelian group G. This convolution has another interpretation as multiplication of polynomials modulo polynomial $(x_0^{m_0}-1)$, $(x_1^{m_1}-1)$, ., $(x_{r-1}^{m_{r-1}}-1)$ when i^{th} element, X^i, of G is indexed as in (2.2.1).

As a consequence, cyclic convolution defined by (2.1.6) and generalized convolution defined by (2.1.3) can be interpreted as

$$b(Z) = a(Z)s(Z) \bmod (Z^n-1) \tag{2.2.6}$$

where Z is the generator of cyclic group C of order n, and

$$b(X) = a(X)s(X) \bmod (x_0^{m_0}-1), (x_1^{m_1}-1), ., (x_{r-1}^{m_{r-1}}-1) \tag{2.2.7}$$

respectively.

Remark 2.2.2: From (2.2.3) we see that group algebra, RG, has the structure of a vector space over $\mathbb{R}$ of dimension n equal to the order of the group G, denoted by $\mathbb{R}^n$. Group algebra can be considered as a ring with addition and multiplication defined as in (2.2.3) and (2.2.4) respectively. Moreover it is a commutative ring with identity.

We now proceed to discuss the notion of Ideals in group algebra.

Definition 2.2.2: A subset I of group algebra,RG,is called an Ideal if

a) I is a linear subspace of $\mathbb{R}^n$

b) For all $a(X) \in I$ and $X \in G$, $Xa(X)$ should be in I.

In the following discussion,some of the results from semi–simple group–algebra which are having relevance to our context will be presented [7].

The group algebra RG over the field of real–numbers is semi–simple group algebra. From Remark (2.2.2) group algebra RG can be viewed as a ring (commutative ring with identity). Semi–simple group algebra RG is a principal ideal ring ; that is every ideal, I, is of the form

$$I = \{s(X)a(X), s(X) \in RG\} \text{ for some element } a(X) \in RG.$$

An ideal I generated by s(X) is denoted by $< s(X) >$. In semi–simple group algebra,

every ideal has an idempotent generator $\theta(X)$. This idempotent generator has the following properties.

$$\text{Let } \theta(X) = \sum_{i=0}^{n-1} \theta_X X^i \tag{2.2.8}$$

$$[\theta(X)]^2 = \theta(X)$$

$$s(X) \in I \Longleftrightarrow s(X)\theta(X) = s(X) \tag{2.2.9}$$

It is also known from the theory of semi–simple group algebras that RG is the direct sum of its minimal ideals. A minimal ideal is one which has no proper sub–ideals.

$$RG = < M_0 > \oplus < M_1 > \oplus \ldots \oplus < M_{t-1} > \tag{2.2.10}$$

Any other ideal in RG can be expressed as a direct sum of subset of these minimal ideals.

The idempotent generators $\theta_i(X)$ of M_i satisfy the following properties:

$$\sum_{i=0}^{t-1} \theta_i(X) = 1 \tag{2.2.11}$$

$$\theta_i(X)\theta_j(X) = 0 \quad ; i \neq j \tag{2.2.12}$$

$$< \theta_i(X) > \cap < \theta_j(X) > = 0 \quad ; i \neq j \tag{2.2.13}$$

The results from the group theory that any finite group of order n is isomorphic to a regular permutation group of the same degree and that a transitive abelian permutation group is a regular permutation group, lead us to see the connection between ideal and the output signal space of the system defined by (2.1.3).

In the light of (2.2.4), condition(b) of definition (2.2.2) can also be interpreted as the set of formal sums, formed by the sequences resulting from the application of permutations belonging to PG to $a = (a_0\ a_1\ a_2\ ..\ a_j\ ..\ a_{n-1})$, also should belong to I. Therefore, output signal space of the system defined by (2.1.3) can also be viewed as Ideal in the group algebra RG. In the following theorem, we prove that there is one to one correspondence between ideals in group algebra and the subspaces invariant relative to PG.

Theorem 2.2.1: There is one–to–one correspondence between subspaces invariant relative to PG over $\mathbb{R}$ and the ideals in the group algebra, RG of abelian group G.

Proof: Let $\mathbb{R}_r$ be a linear subspace of $\mathbb{R}^n$ and RG_r denote subspace of RG. Let $a = (a_0\ a_1 .. a_k .. a_i .. a_{i+k} .. a_{n-1})$, be a vector of length n belonging to $\mathbb{R}^n$. Consider $a(X) = \sum_{i=0}^{n-1} a_i\left(\prod_{\alpha=0}^{r-1} x_\alpha^{i_\alpha} \right)$ as the element in group algebra RG corresponding to a. It is obvious that this association is an isomorphism between RG and $\mathbb{R}^n$, considered as vector spaces over $\mathbb{R}$. Hence $\mathbb{R}_r = RG_r$. An element p_k, $k \in Z_n$, permutes a into

$$a = (a_0\ \ a_1\ .. a_k .. a_i\ .. a_{i+k} .. a_{n-1})$$
$$p_k(a) = (a_{0\oplus k}\ a_{1\oplus k} .. a_0 .. a_{i\oplus k} .. a_i\ .. a_{n-1\oplus k})$$

The corresponding element of RG is

$$p_k(a(X)) = \sum_{i=0}^{n-1} a_i X^{i+k} \tag{2.2.14}$$

Since $X^{i+k} = X^i X^k$, (2.2.14) can be written as

$$p_k(a(X)) = \sum_{i=0}^{n-1} a_i X^i X^k = X^k(a(X)) \ ; \ X^k \in G \tag{2.2.15}$$

If $\mathbb{R}_r$ is assumed to be invariant relative to PG,then $p_k(a) \in \mathbb{R}_r$ whenever $a \in \mathbb{R}_r$. Then according to (2.2.15)

$$a(X) \in RG_r,\ X^i \in G \Rightarrow X^i a(X) \in RG_r$$

and therefore RG_r is an ideal in RG.□

With the correspondence between ideals and P–I subspaces closed under the permutations of PG, being thus established, we now move on to define P–I filter in terms of ideals in group algebra.

2.3 Interpretation of a Class of P–I Filters as an Ideal in Group Algebra

From the last section it is known that group–algebra can be expressed as a direct sum of minimal ideals and that any ideal is a direct sum of subset of these minimal ideals. Further, there is one–to–one correspondence between the ideals and the permutation–invariant subspaces. By making use of these facts, $\mathbb{R}^n$ can be decomposed into

$\mathbb{R}_i$s as $\mathbb{R}^n = \mathbb{R}_0 \oplus \mathbb{R}_1 \oplus .. \oplus \mathbb{R}_{t-1}$, where $\mathbb{R}_i$ is the smallest subspace, closed under the permutations of PG, corresponding to the minimal ideal M_i in RG [5]. Any n–tuple, a, of $\mathbb{R}^n$ is uniquely expressed as direct sum of n–tuples, a^0, a^1, .., a^{t-1} belonging to $\mathbb{R}_0$, $\mathbb{R}_1$, .., $\mathbb{R}_{t-1}$ respectively. Since the system, T, is defined with respect to PG the subspaces $\mathbb{R}_i$ s are invariant under T also. The fact that each $\mathbb{R}_i$ is invariant under T enables us to view the action of T as the independent action of transformations T_i on the subspaces $\mathbb{R}_i$, where T_i is the transformation T restricted to $\mathbb{R}_i$. Therefore the action of T on $a \in \mathbb{R}^n$ can be expressed in the following form.

$$Ta = T_0 a^0 \oplus T_1 a^1 \oplus .. \oplus T_{t-1} a^{t-1} \tag{2.3.1}$$

In other words, this T can be described as the direct sum of the transformations T_0 T_1, .., T_{t-1}. Our purpose is to study T by finding invariant direct–sum decompositions in which the transformations T_i are of elementary nature. Towards this end, each class of these elementary transformations can be interpreted as a minimal ideal in the group algebra for which well–known methods of characterizations are available in the literature. We now give the definition of a P–I filter in group algebra as follows.

Definition 2.2.3: P–I filter is a system, T, which maps any element of group algebra RG into an element of an ideal in the group–algebra and its input–output relationship is given by

$$b(X) = s(X)\, a(X) \bmod (X_0^{m_0} - 1), (X_1^{m_1} - 1), ., (X_{r-1}^{m_{r-1}} - 1) \tag{2.3.2}$$

where a(X), s(X) and b(X) are input, unit–sample response and output elements of the filter.

For the case of cyclic P–I filter this relation takes the form of

$$b(Z) = s(Z)\, a(Z) \bmod (Z^n - 1) \tag{2.3.3}$$

where Z is the generator of cyclic group of order n.

Remark 2.3.1: The unit–sample response element s(X) of T can be interpreted as the

direct sum of vectors belonging to the minimal ideals in the group–algebra. Output space of T is the ideal generated by s(X).

Remark 2.3.2: A set of P–I filters that maps elements of RG into the same ideal forms an ideal in group algebra. This follows from the definition of the ideal. We call this set as an ideal of P–I filters. This affords a classification of P–I filters based on the ideals in group algebra RG. Consequently, a class of cyclic P–I filters that maps into the same ideal can be interpreted as an ideal in cyclic group algebra RC and abelian P–I filters as an ideal in abelian group algebra RG.

A P–I filter is said to be *minimal P–I filter* if its output signal space does not contain any nontrivial sub ideals or if the ideal generated by s(X) is a minimal ideal. A class of P–I filters, {s(x)}, that generate a common minimal ideal is a minimal ideal in RG. Similarly a class of cyclic P–I filters, {s(Z)}, that map into a common minimal ideal is a minimal ideal in cyclic group algebra RC. To distinguish the ideals in RG from the ideals in RC, we call the former as abelian ideals and the latter as cyclic ideals.

Remark 2.3.3: Some of the advantages of classification of P–I filters based on the ideals of group algebra are as follows.

1) This is concerned with the implementation of a P–I filter. In the beginning of this section, we have shown that a P–I filter can be thought of as the direct sum of minimal P–I filters. It is known from the work of T.Beth [8] that it is more efficient to work out computations in each summand minimal ideal than in the group algebra itself. This can be made use of in implementing a P–I filter.

2) In order to characterize a P–I filter in terms of a linearity property say 't', sometimes it will be advantageous to characterize an ideal corresponding to that P–I filter in terms of 't'. This will be made use of while studying cyclically closed abelian P–I filters in the next chapter.

In the following section, we present the transform domain characterization of ideals in group algebra which means the characterization of P–I filters.

2.4 Brief Review of Transform Domain Characterization of Ideals

From Section 2.2, it is known that every ideal in RG is characterized by an idempotent generator. Therefore, by characterizing the minimal idempotent generators in RG, minimal ideals in RG can be characterized. Characterization of these Idempotent generators is more easy in the transform domain rather than in the sample domain.

Generalized Walsh–Hadamard transform defined with respect to the mixed–radix set $\{m_0, m_1, .., m_{r-1}\}$ maps $a = (a_0\ a_1\ a_2\ ..\ a_j\ ..\ a_{n-1}) \in \mathbb{R}^n$ into a complex vector $A = (A_0\ A_1\ A_2\ ..\ A_i\ ..\ A_{n-1})$ in the transform domain. These two vectors are related by the following equations.

$$A_i = \sum_{j=0}^{n-1} \left(\prod_{\alpha=0}^{r-1} \gamma_{m_\alpha}^{-i_\alpha j_\alpha} \right) a_j \quad ; \quad i \in Z_n \quad \text{and} \quad \gamma_{m_\alpha} \text{ is } m_\alpha^{th} \text{ root of unity.} \tag{2.4.1}$$

$$a_j = \sum_{i=0}^{n-1} \left(\prod_{\alpha=0}^{r-1} \gamma_{m_\alpha}^{+i_\alpha j_\alpha} \right) A_i \quad ; j \in Z_n \tag{2.4.2}$$

Equations (2.4.1) and (2.4.2) are known as forward and inverse transforms respectively.They form Generalized Walsh–Hadamard transform pair.

If the sequence a is from $\mathbb{R}^n$, then the samples of the transform sequence A satisfy the following relationship, generally known as *complex–conjugacy relation* between the transform coefficients.

$$A_i = A^{*}_{n\Theta i} \quad ; \quad * \text{ denotes complex–conjugate} \tag{2.4.3}$$

where $n\Theta i = < (m_0 - i_0)_{m_0}\ (m_1 - i_1)_{m_1}\ ..\ (m_{r-1} - i_{r-1})_{m_{r-1}} >$

The set of transform coefficients $\{ A_i, A_{n\Theta i} \}$ is called a conjugacy class. A conjugacy class consists of a single real–valued element whenever $i = n \Theta i$.

Let $\mathbb{C}^n_r$ denote the complex space restricted to the real images. It can be shown that GWHT is an isomorphism between $\mathbb{R}^n$ and $\mathbb{C}^n_r$.

Theorem 2.4.1: Generalized Walsh–Hadamard transform is an isomorphism between the space of n–tuples over R and the subspace, $\mathbb{C}^n_r$, of complex space, $\mathbb{C}^n$, restricted to the real

images.

Proof: Proof follows from the properties of GWHT.□

Since GWHT is an isomorphism between $\mathbb{R}^n$ and $\mathbb{C}_r^n$, the algebras of group G over $\mathbb{R}$ and $\mathbb{C}_r^n$ are also isomorphic. Let us denote the group algebra of G over complex field, $\mathbb{C}_r^n$ by CG_r. Transform vector $A \in \mathbb{C}_r^n$ is the corresponding element A(X) of CG_r, where $A(X) = \sum_{i=0}^{n-1} A_i(\prod_{\alpha=0}^{r-1} X_\alpha^{i_\alpha})$ where $A_i \in \mathbb{C}_r^n$ and $X_\alpha^{i_\alpha} \in C_{m_\alpha}$, a cyclic group of order m_α with generator X_α.

We now proceed to discuss the concept of ideal in CG_r [6].

Theorem 2.4.2: In the transform domain, the set of elements (polynomials) of CG_r having a common set of coefficients which are zero constitute an ideal in CG_r.

Proof: Let I be the set of polynomials of CG_r which have $i_1, i_2, .., i_s^{th}$ coefficients as zero, where $i_1, i_2, .., i_s \in Z_n$. First we show that I is a subgroup of CG_r under addition.

Let $A(X) = \sum_{j=0}^{n-1} A_j(\prod_{\alpha=0}^{r-1} X_\alpha^{j_\alpha})$ and $B(X) = \sum_{j=0}^{n-1} B_j(\prod_{\alpha=0}^{r-1} X_\alpha^{j_\alpha})$ belong to I. i.e., $A_j = B_j = 0$ for $j \in \{i_1, i_2, .., i_s\}$. Also let $C(X) = A(X) + B(X)$ where $C(X) = \sum_{j=0}^{n-1} C_j(\prod_{\alpha=0}^{r-1} X^{j_\alpha})$. For any $j \in \{i_1, i_2, .., i_s\}$, $C_j = A_j + B_j = 0 + 0 = 0$. Hence $C(X) \in I$. i.e., I is closed under addition. Polynomial with all its coefficients as zero is in I which is the identity. The inverse of any element A(X) in I is (−A(X)) which is also in I and associativity axiom is also satisfied. Hence I is subgroup of CG_r under addition.

Now, let $D(X) = \sum_{j=0}^{n-1} D_j(\prod_{\alpha=0}^{r-1} X_\alpha^{j_\alpha})$ be an arbitrary element in CG_r. Also let $C(X) = \sum_{j=0}^{n-1} C_j(\prod_{\alpha=0}^{r-1} X_\alpha^{j_\alpha}) = \sum_{j=0}^{n-1} A_jD_j(\prod_{\alpha=0}^{r-1} X_\alpha^{j_\alpha})$. Since $A(X) \in I$, $A_j = 0$ for all $j \in \{i_1, i_2, .., i_s\}$, which means $A_j.D_j = 0$ for all $j \in \{i_1, i_2, .., i_s\}$. i.e. $C_j = 0$ for all $j \in \{i_1, i_2, .., i_s\}$. i.e. $C(X) \in I$. Hence I is an ideal in CG_r.□

We now give the equivalent description of minimal ideal, idempotent and minimal idempotent in the transform–domain

Minimal ideal: The set of elements of CG_r having a common set of coefficients which are non–zero over a single conjugacy class forms a minimal ideal in CG_r.

Idempotent: An element $E(X) = \sum_{j=0}^{n-1} E_j(\prod_{\alpha=0}^{r-1} X_\alpha^{j_\alpha}) \in CG_r$ is called an idempotent if it satisfies the following relationship.

$$E_j.E_j = E_j \tag{2.4.4}$$

Minimal idempotent: An element of CG_r is having its coefficients as 1 s over a single conjugacy class and all other coefficients as 0 s is a minimal idempotent generator.

Let t be the number of conjugacy classes in CG_r. Denote the element of CG_r which is having its coefficients as 1 s over an i^{th} conjugacy class and other coefficients as 0 s by $E^i(X)$. This $E^i(X)$ is nothing but the i^{th} minimal idempotent generator of the i^{th} minimal ideal, $< E^i(X) >$. There will be as many minimal ideals as the number of conjugacy classes. Therefore CG_r can be expressed in the following form as the direct sum of minimal ideals.

$$CG_r = < E^0(X) > \oplus < E^1(X) > \oplus .. \oplus < E^{t-1}(X) > \tag{2.4.5}$$

Any other ideal in CG_r can be expressed as a direct sum of subset of these minimal ideals.

The idempotent generators $E^i(X)$ of $< E^i(X) >$ satisfy the following properties:

$$\sum_{i=0}^{t-1} E_j^i = 1 \; ; \; j \in Z_n \tag{2.4.6}$$

$$E_k^i E_k^j = 0 \quad ; i \neq j, k \in Z_n \tag{2.4.7}$$

$$< E^i(X) > \cap < E^j(X) > = 0 \quad ; i \neq j \tag{2.4.8}$$

Before giving examples to illustrate above concepts, we digress to a discussion of the expression for the number of conjugacy classes in different real group algebras.

Number of conjugacy classes in CG_r:

For the purpose of calculating the number of conjugacy classes, we shall introduce the following artifice which will be helpful. It should be noted that the length of the

conjugacy class is either 1 or 2. If a group algebra has s single element conjugacy classes and d two element conjugacy classes, we shall represent the expression for the number of conjugacy classes as (s S + d D). The total number can be obtained by putting S = 1 and D = 1. If there are two or more cyclic groups, then the number of conjugacy classes is obtained as follows:

1) Symbolically multiply the expressions corresponding cyclic groups as polynomials in S and D.

2) Evaluate the above polynomial in S and D using the following rules.

$$S.S = S.D = D.S = 1 \; ; \; D^r = 2^{r-1} \; ; \; S^i D^j = D^j = 2^{j-1}.$$

The reason for the above rule is that the product of two cyclic conjugacy classes both of length 2 produces two conjugate classes of abelian group algebra each of length 2. In the rest of the cases only one conjugate class will be produced and the length of the conjugacy class is the product of the lengths of individual cyclic conjugate classes. In the following we give closed form expressions for some cases.

1) When G is a cyclic group of order n

Number of conjugacy classes (t) = (2S + ((n/2)–1)D) for n even

(2.4.9)

$$= (S + ((n-1)/2)D) \text{ for n odd.}$$

(2.4.10)

As explained earlier S and D refers to the conjugacy classes of lengths one and two respectively. For example, for n = 9, there are 5 conjugacy classes out of which one is of length one and the rest four are of length 2.

2) When G is a direct product group, $C_{m_0} \otimes C_{m_1} \otimes . \otimes C_{m_{r-1}}$, of order n

Number of conjugacy classes (t) = $\prod_{i=0}^{r-1} t_i$ (2.4.11)

where t_i is the number of conjugacy classes of a component cyclic group C_{m_i}. In this product substitute 1 for D.S = S.D = S.S and 2 for D.D (In general $D^r = 2^{r-1}$). That gives

the number of conjugacy classes in CG_r . To illustrate this consider the following example.

Example 2.4.1: For $G = C_3 \oplus C_3$, $t = ?$

$$t = t_0.t_1 = (S + D).(S + D) = S.S + 2\,S.D + D.D = 1 + 2 + 2 = 5.$$

The following interesting theorem may have an important consequence in regard to the study of filters.

Theorem 2.4.3: For odd m, the number of conjugacy classes in group algebra CC_r of a cyclic group of order m^p, is same as the number of conjugacy classes in group algebra CG_r of group G where G is a direct product group of p cyclic subgroups, each of order m.

Proof: Let t_c and t_a denote the number of conjugacy classes in CC_r and CG_r respectively. Using (2.4.10), $t_c = (m^p + 1)/2$. Let t be the number of conjugacy classes of C_m algebra. Using (2.4.11) and binomial expansion,

$$t_a = [S + ((m-1)/2)D]^p = \sum_{j=0}^{p} {}^pC_j\, S^{p-j}\,(m'D)^j \quad ; m' = (m-1)/2$$

Separating the term corresponding to j = 0 and invoking the rules of evaluation, the above expression simplifies to

$$t_a = 1 + \sum_{j=1}^{p} {}^pC_j\, S^{p-j}\,(m'D)^j = 1 + \sum_{j=1}^{p} {}^pC_j\,(m'D)^j = 1 + (1/2)\sum_{j=1}^{p} {}^pC_j\,(m-1)^j$$

Since using binomial theorem $\sum_{j=0}^{p} {}^pC_j\,(m-1)^j = m^p$, t_a is given by

$$t_a = 1 + (m^p-1)/2 = (m^p + 1)/2 = t_c \text{ as required.} \square$$

In what follows, we consider some examples and give the characterization of ideals in the form of tables. It is to be noted that ideals in cyclic group algebra CC_r are expressed as polynomials in Z.

Example 2.4.2: $n = 8$; $r = 1$; $t = 5$; $Z^8 = 1$

Table 2.4.1: List of Non–zero Conjugacy Classes along with their Minimal Idempotents and the General Form of Elements of the Corresponding Minimal Ideals in C_8 Group Algebra.

Conjugacy class no. i	Minimal idempotent $E^i(Z)$	Form of elements of ideal generated by $E^i(Z)$
0	$E^0(Z)=1$	$á_0$
1	$E^1(Z)=Z + Z^7$	$A_1Z + A_1^*Z^7$
2	$E^2(Z)=Z^2 + Z^6$	$A_2Z^2 + A_2^*Z^6$
3	$E^3(Z)=Z^3 + Z^5$	$A_3Z^3 + A_3^*Z^5$
4	$E^4(Z)=Z^4$	$á_4Z^4$

Example 2.4.3: $n=8$; $r=2$; $m_0=4$, $m_1=2$; $t=6$; $X_0^4 = X_1^2 =1$

Table 2.4.2: List of Non–zero Conjugacy Classes along with their Minimal Idempotents and the General Form of Elements of the Corresponding Minimal Ideals in $C_4 \oplus C_2$ Group algebra.

Conjugacy class no. i	Minimal idempotent $E^i(X)$	Form of elements of ideal generated by $E^i(X)$
0	$E^0(X)=1$	$á_0$
1	$E^1(X)=X_0 + X_0^3$	$A_1X_0 + A_1^*X_0^3$
2	$E^2(X)=X_0^2$	$á_2X_0^2$
3	$E^3(X)=X_1$	$á_3X_1$
4	$E^4(X)=X_1X_0+X_1X_0^3$	$A_4X_1X_0+A_4^*X_1X_0^3$
5	$E^4(X)=X_1X_0^2$	$á_5X_1X_0^2$

Example 2.4.4: $n=32$; $r=1$; $t=17$; $Z^{32}=1$

Table 2.4.3: List of Non–zero Conjugacy Classes along with their Minimal Idempotents and the General Form of Elements of the Corresponding Minimal Ideals in C_{32} Group Algebra.

Conjugacy class no. i	Minimal idempotent $E^i(Z)$	Form of elements of ideal generated by $E^i(Z)$
0	$E^0(Z)=1$	a'_0
1	$E^1(Z)=Z+Z^{31}$	$A_1 Z + A^*_1 Z^{31}$
2	$E^2(Z)=Z^2+Z^{30}$	$A_2 Z^2 + A^*_2 Z^{30}$
3	$E^3(Z)=Z^3+Z^{29}$	$A_3 Z^3 + A^*_3 Z^{29}$
4	$E^4(Z)=Z^4+Z^{28}$	$A_4 Z^4 + A^*_4 Z^{28}$
5	$E^5(Z)=Z^5+Z^{27}$	$A_5 Z^5 + A^*_1 Z^{27}$
6	$E^6(Z)=Z^6+Z^{26}$	$A_6 Z^6 + A^*_6 Z^{26}$
7	$E^7(Z)=Z^7+Z^{25}$	$A_7 Z^7 + A^*_7 Z^{25}$
8	$E^8(Z)=Z^8+Z^{24}$	$A_8 Z^8 + A^*_8 Z^{24}$
9	$E^9(Z)=Z^9+Z^{23}$	$A_9 Z^9 + A^*_9 Z^{23}$
10	$E^{10}(Z)=Z^{10}+Z^{22}$	$A_{10} Z^{10} + A^*_{10} Z^{22}$
11	$E^{11}(Z)=Z^{11}+Z^{21}$	$A_{11} Z^{11} + A^*_{11} Z^{21}$
12	$E^{12}(Z)=Z^{12}+Z^{20}$	$A_{12} Z^{12} + A^*_{12} Z^{20}$
13	$E^{13}(Z)=Z^{13}+Z^{19}$	$A_{13} Z^{13} + A^*_{13} Z^{19}$
14	$E^{14}(Z)=Z^{14}+Z^{18}$	$A_{14} Z^{14} + A^*_{14} Z^{18}$
15	$E^{15}(Z)=Z^{15}+Z^{17}$	$A_{15} Z^{15} + A^*_{15} Z^{17}$
16	$E^{16}(Z)=Z^{16}$	$a'_{16} Z^{16}$

xample 2.4.5: $n=32$; $r=2$; $m_0=4$, $m_1=8$; $t=18$; $X_0^4 = X_1^8 = 1$

Table 2.4.1: List of Non–zero Conjugacy Classes along with their Minimal Idempotents and the General Form of Elements of the Corresponding Minimal Ideals in $C_4 \otimes C_8$ Group Algebra.

Conjugacy class no. i	Minimal idempotent $E^i(X)$	Form of elements of ideal $< E^i(X) >$
0	$E^0(X)=1$	a'_0
1	$E^1(X)=X_0+X_0^3$	$A_1X_0+A_1^*X_0^3$
2	$E^2(X)=X_0^2$	$a'_2X_0^2$
3	$E^3(X)=X_1+X_1^7$	$A_3X_1+A_3^*X_1^7$
4	$E^4(X)=X_1X_0+X_1^7X_0^3$	$A_4X_1X_0+A_4^*X_1^7X_0^3$
5	$E^5(X)=X_1X_0^2+X_1^7X_0^2$	$A_5X_1X_0^2+A_5^*X_1^7X_0^2$
6	$E^6(X)=X_1^2+X_1^6$	$A_6X_1^2+A_6^*X_1^6$
7	$E^7(X)=X_1^2X_0+X_1^6X_0^3$	$A_7X_1^2X_0+A_7^*X_1^6X_0^3$
8	$E^8(X)=X_1^2X_0^2+X_1^6X_0^2$	$A_8X_1^2X_0^2+A_8^*X_1^6X_0^2$
9	$E^9(X)=X_1^3+X_1^5$	$A_9X_1^3+A_9^*X_1^5$
10	$E^{10}(X)=X_1^3X_0+X_1^5X_0^3$	$A_{10}X_1^3X_0+A_{10}^*X_1^5X_0^3$
11	$E^{11}(X)=X_1^3X_0^2+X_1^5X_0^2$	$A_{11}X_1^3X_0^2+A_{11}^*X_1^5X_0^2$
12	$E^{12}(X)=X_1^4$	$a'_{12}X_1^4$
13	$E^{13}(X)=X_1^4X_0+X_1^4X_0^3$	$A_{13}X_1^4X_0+A_{13}^*X_1^4X_0^3$
14	$E^{14}(X)=X_1^4X_0^2$	$a'_{14}X_1^4X_0^2$
15	$E^{15}(X)=X_1^5X_0+X_1^3X_0^3$	$A_{15}X_1^5X_0+A_{15}^*X_1^3X_0^3$
16	$E^{16}(X)=X_1^6X_0+X_1^2X_0^3$	$A_{16}X_1^6X_0+A_{16}^*X_1^2X_0^3$
17	$E^{17}(X)=X_1^7X_0+X_1X_0^3$	$A_{17}X_1^7X_0+A_{17}^*X_1X_0^3$

2.5 Transform Domain Classification of P–I Filters

In this section, we present the transform–domain classification of abelian P–I filters and cyclic P–I filters based on ideals in their respective group algebras. First we start with abelian P–I filters.

In the transform–domain, abelian P–I filter modifies the spectrum of an input signal as given below

$$B_i = S_i \cdot A_i \tag{2.5.1}$$

where $B \underset{\text{FGWHT}}{\xrightarrow{\text{IGWHT}}} b$, $S \underset{\text{FGWHT}}{\xrightarrow{\text{IGWHT}}} s$ and $A \underset{\text{FGWHT}}{\xrightarrow{\text{IGWHT}}} a$.

Any abelian P–I filter vector, $S \in C_r^n$ can be interpreted as a linear combination of vectors which are non–zero over a single conjugacy class. Let S^i denote a vector which is non–zero over an i^{th} conjugacy class. As there are t conjugacy classes S can be represented as

$$S = S^0 + S^1 + \dots + S^{t-1} \tag{2.5.2}$$

These S^i are nothing but the elements of minimal ideals generated by $E^i(X)$. Therefore, transform of an abelian P–I filter vector can be interpreted as direct sum of component vectors of minimal ideals and the set of abelian P–I filters whose transform vectors have zeros at a common set of locations forms an ideal in abelian group algebra.

Remark 2.5.1: If the transform vector of an abelian P–I filter is a binary valued (0 or 1) vector then that abelian P–I filter can be interpreted as an idempotent.

Example 2.5.1: $n = 8$; $r = 2$; $m_0 = 4$, $m_1 = 2$, $t = 6$

General form of abelian P–I filter vector is $S = (a'_0 \ A_1 \ a'_2 \ A_1^* \ a'_3 \ A_4 \ a'_5 \ A_4^*)$

Let $S = S^0 + S^1 + S^2 + S^3 + S^4 + S^5$ where

$S^0 = (a'_0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)$; $S^0(X) = a'_0$

$S^1 = (0 \ A_1 \ 0 \ A_1^* \ 0 \ 0 \ 0 \ 0)$; $S^1(X) = A_1X_0 + A_1^*X_0^3$

$S^2 = (0\ 0\ \acute{a}_2\ 0\ 0\ 0\ 0\ 0)$; $S^2(X) = \acute{a}_2 X_0^2$
$S^3 = (0\ 0\ 0\ 0\ \acute{a}_3\ 0\ 0\ 0)$; $S^3(X) = \acute{a}_3 X_1$
$S^4 = (0\ 0\ 0\ 0\ 0\ A_4\ 0\ A_4^*)$; $S^4(X) = A_4 X_1 X_0 + A_4^* X_1 X_0^3$
$S^5 = (0\ 0\ 0\ 0\ 0\ 0\ \acute{a}_5\ 0)$; $S^5(X) = \acute{a}_5 X_1 X_0^2$

$S^i(X)$ is a general element of the minimal ideal generated by an i^{th} idempotent generator.

For this example, we now illustrate how an abelian P–I filter can be interpreted as an idempotent. The minimal idempotent generators are

$E^0=(1\,0\,0\,0\,0\,0\,0\,0\,0)$

$E^1=(0\,1\,0\,1\,0\,0\,0\,0\,0)$

$E^2=(0\,0\,1\,0\,0\,0\,0\,0\,0)$

$E^3=(0\,0\,0\,0\,1\,0\,0\,0\,0)$

$E^4=(0\,0\,0\,0\,0\,0\,1\,0\,1)$

$E^5=(0\,0\,0\,0\,0\,0\,0\,1\,0)$.

Any binary valued transform vector of an abelian P–I filter is a combination of E^i.

We now take up the classification of cyclic P–I filters based on ideals in cyclic group algebra. In the transform–domain, a cyclic P–I filter modifies the DFT spectrum of an input signal in accordance with the following equation

$$B_i = S_i \cdot A_i \qquad (2.5.3)$$

where $B \underset{\text{FDFT}}{\overset{\text{IDFT}}{\rightleftarrows}} b$, $S \underset{\text{FDFT}}{\overset{\text{IDFT}}{\rightleftarrows}} s$ and $A \underset{\text{FDFT}}{\overset{\text{IDFT}}{\rightleftarrows}} a$

The filter vector can be interpreted as direct sum of component vectors of the minimal ideals of cyclic group algebra CC_r and the set of cyclic filters whose DFT vectors have zero–valued components at a common set of locations is an ideal. Here also, a cyclic filter whose transform vector is a binary valued vector is an idempotent.

Example 2.5.2: $n = 8$; $r = 1$; $t = 5$

General form of cyclic P–I filter vector is $S = (\acute{a}_0\ A_1\ A_2\ A_3\ \acute{a}_4\ A_3^*\ A_2^*\ A_1^*)$

Let $S = S^0 + S^1 + S^2 + S^3 + S^4$ where

$S^0 = (\acute{a}_0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)$; $S^0(Z) = \acute{a}_0$

$S^1 = (0\ A_1\ 0\ 0\ 0\ 0\ 0\ A_1^*)$; $S^1(Z) = A_1Z + A_7^*Z^7$

$S^2 = (0\ 0\ A_2\ 0\ 0\ 0\ A_2^*\ 0)$; $S^2(Z) = A_2Z^2 + A_2^*Z^6$

$S^3 = (0\ 0\ 0\ A_3\ 0\ A_3^*\ 0\ 0)$; $S^3(Z) = A_3Z^3 + A_3^*Z^5$

$S^4 = (0\ 0\ 0\ 0\ \acute{a}_4\ 0\ 0\ 0)$; $S^4(Z) = \acute{a}_4Z^4$

$S^i(Z)$ is a general element of the minimal ideal generated by i^{th} idempotent generator.

For this example, we now illustrate how a cyclic P–I filter can be interpreted as an idempotent. The minimal idempotent generators are

$E^0 = (1\,0\,0\,0\,0\,0\,0\,0)$

$E^1 = (0\,1\,0\,0\,0\,0\,0\,1)$

$E^2 = (0\,0\,1\,0\,0\,0\,1\,0)$

$E^3 = (0\,0\,0\,1\,0\,1\,0\,0)$

$E^4 = (0\,0\,0\,0\,1\,0\,0\,0)$

Any binary valued transform vector of a cyclic P–I filter is a combination of E^i.

CHAPTER 3
CHARACTERIZATION OF CYCLICALLY CLOSED ABELIAN P-I FILTERS

So far, we have discussed classification of P–I filters based on the ideals in group algebra. Consequently, cyclic P–I filters belong to the ideals in cyclic group algebra and abelian P–I filters belong to the ideals in abelian group algebra. Here, we present an important work of this thesis. We consider the characterization of P–I filters that have the structures of both cyclic as well as abelian filters inherently. Such P–I filters are closed under both cyclic and abelian permutations. We call this type of P–I filters as cyclically closed abelian P–I filters. MRX mapping between abelian and cyclic group algebras provides a basis for the characterization of these filters. Whenever the dimension of a P–I filter is a product of pairwise relatively prime integers, we consider the mapping based on CRT also and show that under this mapping any cyclic P–I filter can be realized equivalently as an abelian P–I filter. In what follows, we discuss the significance of this study in the implementation of cyclic P–I filters.

Concurrency or parallelism inherent to the structure of abelian class of P–I filters is more than that of cyclic class of P–I filters [1]. Therefore this can be exploited in the abelian P–I filter implementation of a cyclic P–I filter, in applications such as real–time signal processing where high computational throughput is required. And also, the transform–domain implementation of a cyclic P–I filter using GWHT requires lesser memory space than that is required for DFT. For example, to implement a cyclic P–I filter of length n, if we use DFT it requires n^2 memory locations for storing transform matrix, whereas, GWHT, with factors m_i requires $\Sigma\, m_i^2$ memory locations which is far lesser than that of DFT. Apart from these two advantages this study answered the questions like, by

using GWHT, is it possible to extract all the features those can be extracted by using DFT ? , if not all, then what are those cyclic features that GWHT can extract ? .

The problem is approached as follows. For a given class of abelian P–I filters, we characterize all the elements of abelian group algebra whose cyclic permutations are same as the corresponding abelian permutations. This correspondence is established on the basis of MRX mapping. Because of the necessity that these elements should belong to the ideals of abelian group algebra that are closed under cyclic permutations, firstly, we completely characterize all the ideals that are closed under cyclic permutations in the abelian group algebra. The converse of the above that whether any element of a cyclically closed abelian ideal is also a cyclically closed abelian P–I filter, may not be true always. This fact will be illustrated later with reasons. Secondly, since all the elements of cyclically closed ideals in abelian group algebra need not have the property that the effect of cyclic permutations are same as that of abelian permutations, we characterize all those elements which are cyclically closed abelian P–I filters.

The organization of this chapter is as follows. In Section 3.1, we describe cyclically closed abelian P–I filters. In Section 3.2, we interpret cyclically closed abelian P–I filters as elements of cyclically closed ideals in abelian group algebra. In Section 3.3, a theorem concerning the characterization of ideals in abelian group algebra that are closed under cyclic permutations is given. In Section 3.4, characterization of cyclically closed abelian P–I filters of a class of abelian P–I filters is presented in the form of a theorem. In Section 3.5, transform domain characterization of cyclically closed abelian filters of a class of abelian filters is given. In Section 3.6, the notion of a 2–D P–I filter is explained along with a procedure for identifying 2–D cyclically closed abelian filters. Finally in Section 3.7, we consider cyclic mappings based on CRT and show that under this mapping cyclic and abelian group algebras are isomorphic and that a cyclic P–I filter whose dimension is a product of pairwise relatively prime integers can be realized as an equivalent abelian P–I filter.

3.1 Cyclically Closed Abelian P–I Filters

In what follows, the concept of a cyclically closed abelian P–I filter is explained by means of an example.

Let us consider a P–I filter defined with respect to the cyclic permutation group PC_8 of order 8 and another with respect to an abelian permutation group PG_8, which is defined with respect to the mixed–radix set $M = \{ m_0 = 4 , m_{-1} = 2 \}$, of order 8. The former filter is represented by an 8 x 8 matrix S_C and the latter by another 8 x 8 matrix S_A.

Consider a vector $s = (s_0\ s_1\ s_2\ s_3\ s_4\ s_5\ s_6\ s_7) \in \mathbb{R}^8$ to represent zeroth column of S_C and S_A.

Any k^{th} column of S_C, $(k \in Z_8)$, is obtained by applying k^{th} permutation, p_k, of PC_8 to s as given below.

$$p_k(s) = (s_{0 \ominus k}\ s_{1 \ominus k} \cdots s_{i \ominus k} \cdots s_{7 \ominus k}) \tag{3.1.1}$$

where $(i \ominus k)$ is $(i-k)$ modulo 8.

In a similar manner, the k^{th} column of S_A is obtained by applying k^{th} permutation, p_k of PG_8 to s as shown below.

$$p_k(s) = (s_{0 \ominus k}\ s_{1 \ominus k} \cdots s_{i \ominus k} \cdots s_{7 \ominus k}) \tag{3.1.2}$$

where $i \ominus k = < (i_0 - k_0)_4\ \ (i_1 - k_1)_2 >$.

Consider a vector of form $(a_1\ \ 0\ \ 0\ \ 0\ \ a_2\ \ 0\ \ 0\ \ 0) \in \mathbb{R}^8$. For this, we write the P–I filter matrices S_C and S_A, using the above (3.1.1) and (3.1.2).

$$S_A = \begin{bmatrix} a_1 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & a_1 & 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_1 & 0 & 0 & 0 & a_2 \\ a_2 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_2 & 0 & 0 & 0 & a_1 \end{bmatrix}$$

$$S_C = \begin{bmatrix} a_1 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & a_1 & 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_1 & 0 & 0 & 0 & a_2 \\ a_2 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & a_2 & 0 & 0 & 0 & a_1 \end{bmatrix}$$

It can be seen from the above that both S_C and S_A are identical. Therefore, other columns of any abelian P–I filter matrix with the zeroth column of the above type can be obtained by applying permutations belonging to either CC_8 or CG_8 . That means this abelian P–I filter is also closed under the permutations of CC_8 . Hence, this type of filters are called *Cyclically closed abelian P–I filters.* We now formally define cyclically closed abelian P–I system.

Definition 3.1.1: A system, T, which is defined invariant relative to a pair of cyclic transitive abelian permutation group, PC, of degree n and a transitive abelian permutation group, PG, of degree n, whose elements, p_k , $k \in Z_n$, are being specified by the mixed–radix set $\{m_0, m_1,.., m_{r-1}\}$, is characterized by either of the following convolutional relationships.

$$b_i = \sum_{j=0}^{n-1} s_{i\Theta j}\, a_j \tag{3.1.1}$$

where $i\Theta j = < (i_0 - j_0)_{m_0}\ (i_1 - j_1)_{m_1} .. (i_{r-1} - j_{r-1})_{m_{r-1}} >$

Alternatively, (3.1.1) can also be expressed in the form of matrix–equation as

$$b = Sa \tag{3.1.2}$$

where a and b are nx1 matrices and S is nxn matrix. Elements of S are given by

$$S_{i,j} = s_{i\Theta j} \tag{3.1.3}$$

where $i\Theta j$ is as defined above.

$$b_i = \sum_{j=0}^{n-1} s_{(i-j)\bmod n}\, a_j \tag{3.1.4}$$

where $a \in \mathbb{R}^n$ and $s \in \mathbb{R}^n$

Alternatively, (3.1.4) can also be expressed in the form of matrix–equation as

$$b=Sa \tag{3.1.5}$$

where S is nxn matrix whose elements are given by

$$S_{i,j} = s_{(i-j)\bmod n} \tag{3.1.6}$$

Let us denote the system which is defined invariant relative to both cyclic and abelian permutation groups of degree n by T_c. As the system, T_c, maps the space of real n–tuples, $\mathbb{R}^n$, into the space of real n–tuples, in which each of the n–tuples is a linear combination of the column vectors of S, the output signal space of the system T_c is closed under cyclic permutations of PC as well as abelian permutations of PG. Let us denote the subspace of $\mathbb{R}^n$ closed under both cyclic as well as abelian permutations by $\mathbb{R}^n_c$. In this context, the system, T_c, which is a mapping from $\mathbb{R}^n$ into subspace $\mathbb{R}^n_c$ is called a cyclically closed abelian P–I filter. We now formally define cyclically closed abelian P–I filter.

Definition 3.1.2: A cyclically closed abelian P–I filter can be defined as a system T_c, which maps the space of real n–tuples, $\mathbb{R}^n$, into a linear subspace of $\mathbb{R}^n$ which is closed under both cyclic permutations as well as abelian permutations belonging to PG and its input–output relationship is characterized either by (3.1.1) or by (3.1.4).

This cyclically closed abelian P–I filter is said to be *minimally cyclically closed abelian P–I filter* if its output signal space does not contain any proper subspaces closed under both cyclic permutations as well as abelian permutations. Figure 3.1.1 represents the sample domain interpretations of cyclically closed abelian P–I filter.

We now prove a lemma which will be used later.

Lemma 3.1.1: For a cyclically closed abelian P–I filter $S_A - S_C = 0$, where S_C and S_A are the cyclic and abelian P–I matrices associated with the filter.

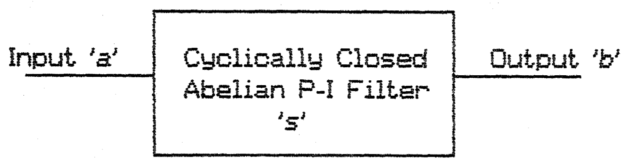

Figure 3.1.1.a

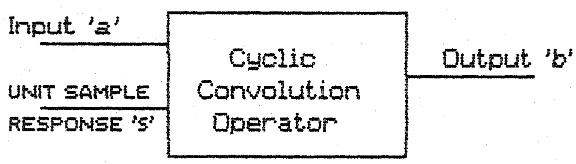

Figure 3.1.1.b

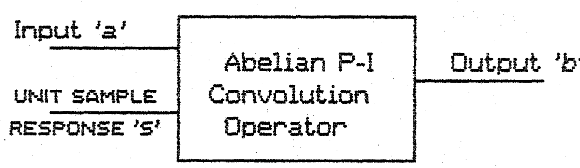

Figure 3.1.1.c

Figure 3.1.1

Sample Domain Interpretations of Cyclically Closed Abelian P-I Filter

Proof: Let s denote the unit–sample response of a cyclically closed abelian P–I filter. Any i,j th element of S_C is $(S_C)_{i,j} = s_{(i-j) \bmod n}$ and that of S_A is $(S_A)_{i,j} = s_{i_\theta j}$. If an abelian P–I filter represented by S_A is invariant relative to PC then, any element of the matrix S_A should be $(S_A)_{i,j} = s_{(i-j) \bmod n}$ which is $(S_C)_{i,j}$. Therefore for a cyclically closed abelian P–I filter $s_{(i-j) \bmod n} = s_{i_\theta j}$ which means that $S_A - S_C = 0$.□

In the following section, we interpret cyclically closed abelian P–I filters as elements of cyclically closed ideals in abelian group algebra.

3.2 Cyclically Closed Abelian P–I Filters as Elements of Cyclically Closed Abelian Ideals

To explain about an ideal in abelian group algebra closed under cyclic permutations first we consider the interpretation of the operation of permutation on a vector as a multiplication of an element of a group algebra by a group element.

Let us consider a vector $a = (a_0\ a_1\ ..\ a_k\ ..\ a_i\ ..\ a_{i+k}\ ..\ a_{n-1}) \in \mathbb{R}^n$. Consider $a(X) = \sum_{i=0}^{n-1} a_i X^i$ as the element corresponding to a in group algebra RG of group G, whose element X^i is indexed with respect to the mixed–radix set $M = \{ m_0\ m_1\ ...\ m_{r-1} \}$ as $X^i = x_{r-1}^{i_{r-1}}\ x_{r-2}^{i_{r-2}}\ ...\ x_1^{i_1}\ x_0^{i_0} = \prod_{\alpha=0}^{r-1} x_\alpha^{i_\alpha}$ where i_α is the index of i with respect to the mixed–radix m_α in the mixed–radix representation of i with respect to the mixed–radix set $M = \{ m_0\ m_1\ ...\ m_{r-1} \}$.

Let PG be the transitive abelian permutation group defined with respect to the mixed–radix set M. Denote an element belonging to PG by p_k.

Now we consider the effect of a permutation belonging to PG on a. An element p_k, $k \in Z_n$, permutes a into

$$a = (a_0\ \ a_1\ ...\ a_k\ ..\ a_i\ ...\ a_{i+k}\ ..\ a_{n-1}) \tag{3.2.1}$$

$$p_k(a) = (a_{0_\theta k}\ a_{1_\theta k}\ ..\ a_0\ ..\ a_{i_\theta k}\ ..\ a_i\ ..\ a_{n-1_\theta k}) \tag{3.2.2}$$

The corresponding element of RG is

$$p_k(a(X)) = \sum_{i=0}^{n-1} a_i X^{i+k} \tag{3.2.3}$$

Since $X^{i+k} = X^i X^k$, (3.2.3) can be written as

$$p_k(a(X)) = \sum_{i=0}^{n-1} a_i X^i X^k = X^k a(X) \quad ; \; X^k \in G. \tag{3.2.4}$$

Therefore permuting an element a with P_k of PG is equivalent to multiplying the corresponding group algebra element with X^k of group G.

After having interpreted permutation as multiplication of a group element, now, we explain about cyclically closed ideal in group algebra as follows.

Cyclic permuted version of a vector can be obtained by treating it as a cyclic group algebra element and then multiplying it with a cyclic group element. Likewise its abelian permuted version can be obtained by treating it as an abelian group algebra element and then multiplying it with an abelian group element. Under mixed–radix (MRX) mapping X^k is mapped into Z^k , where X^k and Z^k are elements of abelian group and cyclic group respectively. If $a(X)$ is an element of an ideal in abelian group algebra RG, then all $X^i a(X)$ belong to this ideal. Now consider $a(Z)$ obtained by applying MRX mapping to $a(X)$. Consider the set of all $Z^k a(Z)$; $k \in Z_n$. This set forms an ideal in cyclic group algebra CG. Therefore this set is a cyclic ideal. Apply inverse MRX mapping to this set. If all the elements of this set belong to the ideal in abelian group algebra, then we call such an ideal as cyclically closed abelian ideal. This ideal is called minimally cyclically closed abelian ideal if it does not contain any sub ideals that are closed under both cyclic as well as abelian permutations. A space of n–tuples closed under both cyclic as well as abelian permutations forms a cyclically closed abelian ideal in a group algebra. For a vector a whose abelian permutation is same as cyclic permutation, multiplying abelian group algebra element, $a(X)$, with X^k is same as multiplying cyclic group algebra element, $a(Z)$ with Z^k.

Since the output signal space of a cyclically closed abelian P–I filter has the

structure of cyclically closed abelian ideal, it can be interpreted in the following ways.

1) A cyclically closed abelian P–I filter is a mapping that maps any element of abelian group algebra RG into an element of a cyclically closed abelian ideal in RG and its input–output relationship is given by

$$b(X) = a(X)\, s(X) \bmod (X_0^{m_0} - 1), (X_1^{m_1} - 1), ., (X_{r-1}^{m_{r-1}} - 1) \tag{3.2.5}$$

where a(X), b(X) and s(X) are input, output and filter's unit–sample response elements respectively.

2) A cyclically closed abelian P–I filter is a mapping that maps any element of cyclic group algebra RC into an element of a cyclically closed abelian ideal in CG and its input–output relationship is given by

$$b(Z) = a(Z)\, s(Z) \bmod (Z^n - 1) \tag{3.2.6}$$

where a(Z), b(Z) and s(Z) are input, output and filter's unit–sample response elements respectively.

The fact that, the effects of cyclic permutations on unit–sample response vectors of cyclically closed abelian P–I filters are same as abelian permutations, these vectors should necessarily belong to cyclically closed abelian ideals. Therefore, cyclically closed abelian P–I filters can be interpreted as elements of cyclically closed abelian ideals. But, the converse that whether an element of a cyclically closed abelian ideal is also a cyclically closed abelian P–I filter, is not always true. This will be explained in Section 3.4. In the following section, we list out all cyclically closed ideals in abelian group algebra in the form of a theorem.

3.3 Identification of Cyclically Closed Ideals in Abelian Group Algebra

In Section 3.2 we have explained the meaning of a cyclically closed ideal in the

abelian group algebra. The relevant mapping is based on a mixed–radix system consisting of mixed radices corresponding to orders of component cyclic groups in the direct product of G (abbreviated MRX mapping). Under this mapping 'g' ε G, an abelian group of order n, is mapped to a corresponding element of C_n that is associated with that particular element of Z_n that corresponds to the mixed–radix digit representation of 'g' and by linearity this notion is extended to MRX mapping of an element of the group algebra. Unlike the CRT mapping (which requires the component cyclic groups orders to be mutually prime) this mapping is possible in any general abelian group algebra. Now we take up the characterization and identification of such cyclic abelian codes under MRX mapping [7].

First we consider an useful lemma based on the linearity of the MRX mapping that shows that given two ideals cyclic under MRX mapping then their sum is also cyclic under MRX mapping. This lemma simplifies the proof considerably in that it is enough to show some basic ideals are cyclic and the remaining cyclic ideals can be obtained as sums of these basic ideals. The lemma is proved for any general linear mapping 't' and therefore in particular for MRX mapping.

Lemma 3.3.1: If C_1 and C_2 are cyclically closed ideals in an abelian group algebra under some linear mapping ' t ', then ' $C_1 + C_2$ ' is also cyclically closed under ' t '.

Proof: Any element of $C_1 + C_2$ is of form a c_1 + b c_2 where $c_1 \in C_1$, $c_2 \in C_2$. Let z(a) denote cyclic shift of *a* by one place. $(a)_i$ denote i^{th} bit of *a*.

Since 't' is a linear mapping

$$t(\, a\, c_1 + b\, c_2\,) = a\, t(\, c_1\,) + b\, t(\, c_2\,)$$

$$(z(t(\, a\, c_1 + b\, c_2\,)))_i = (a\, t(\, c_1\,) + b\, t(\, c_2\,))_{(i-1) \bmod n}$$

$$= (z(a\, t(\, c_1\,)))_i + (z(b\, t(\, c_2\,)))_i$$

By hypothesis z(a t(c_1)) $\in C_1$ and z(b t(c_2)) $\in C_2$. Therefore their sum belongs to $C_1 + C_2$.□

For the purpose of characterization of an abelian cyclic ideal, it is advantageous to introduce the notion of a cyclic ideal being minimally cyclic under MRX mapping. This should not be confused with the notion of a minimal ideal of an abelian group algebra. A minimally cyclic ideal may be a minimal or a non–minimal ideal of the abelian group algebra. This notion facilitates a different way of proving the characterization theorems given in [7]. Using this approach some fallacies in the proof given by [7] can also be taken care of. A formal definition can be given as follows:

Definition 3.3.1: An ideal M is said to be *minimally cyclic* under MRX mapping if it is cyclic and there exists no nontrivial sub ideal N of M that is cyclic under MRX mapping.

One immediate consequence is the following lemma which will be useful later.

Lemma 3.3.2: Suppose RG can be written as a direct sum of the following minimally cyclic ideals as shown below:

$$RG = <M_1> \oplus <M_2> \oplus \ldots \oplus <M_t>$$

Consider two non–minimal ideals, say M_i and M_j. Suppose N_i and N_j be any sub ideals of M_i and M_j respectively. Then $N_i \oplus N_j$ is not cyclic under MRX mapping.

Proof: We prove the lemma by contradiction. Suppose $N_i \oplus N_j$ is cyclic. Since $M_i = N_i \oplus N_k$ is cyclic, it follows from Lemma 3.3.1 that the sum of this ideal with the ideal $N_i \oplus N_j$, i.e., $N_i \oplus N_k \oplus N_j = M_i \oplus N_j$, is cyclic. Since M_i is cyclic, this implies that N_j is cyclic. This contradicts the minimality of M_j. Therefore $N_i \oplus N_j$ can not be cyclic under MRX mapping.□

Remark 3.3.1: From above Lemmas 3.3.1 and 3.3.2 it is enough to find a direct sum decomposition of RG in terms of minimally cyclic ideals in order to characterize all the cyclic ideals in the abelian group algebra. For any nontrivial sub ideal or sum of nontrivial sub ideals of minimally cyclic ideals can not be cyclic according to Lemma 3.3.2. And once minimally cyclic ideals are known, any other cyclic abelian ideal can be obtained as sum of these minimally cyclic ideals according to Lemma 3.3.1.

We shall first consider when G is direct product of two cyclic groups. Once the basic

arguments are clearly understood it is easy to prove the results more generally for the case when G is direct product of r cyclic groups.

Notations: RG denotes an abelian group algebra of group G where $G = C_{m_0} \oplus C_{m_1} \oplus \ldots \oplus C_{m_{r-1}}$; C_{m_i} is a cyclic group of order m_i with generator x_i. We shall represent $\prod_{\alpha=0}^{r-1} x_\alpha^{i_\alpha}$ concisely by X^i, where 'i' is the mixed–radix representation of $< i_0, i_1, \ldots, i_{r-1} >$ w.r.t mixed–radix indices $m_0, m_1, \ldots, m_{r-1}$. Let $z^k[a(X)]$ represent the polynomial corresponding to cyclic shifted vector of a(X) by k places. A minimal idempotent in the RC_{m_i} algebra is denoted by $\theta(x_i)$ and the ideal generated by it is denoted by $<\theta(x_i)>$ and, in particular j^{th} minimal ideal is denoted by $< \theta_j(x_i) >$. Let $R_0(x_i)$ represent the idempotent consisting of all the elements of the group C_{m_i} each with the coefficient 1/n. For example, $R_0(x)$ in RC_3 algebra generated by x, is given by $1/3(1 + x + x^2)$. Number of conjugate classes in the RC_{m_i} algebra is denoted by t_i. $R_0(x_i)$ is taken as t_0^{th} ideal of RC_{m_i} algebra.

Let $M_0 = 1$ and $M_i = \prod_{k=0}^{i-1} m_k$, for k = 1 to r–1. Then MRX to cyclic mapping is given by $x_k \longrightarrow z^{M_k}$, for k = 0 to r–1. In particular, for r = 2 case, MRX to cyclic mapping is given by $x_0 \longrightarrow z$ and $x_1 \longrightarrow z^{m_0}$.

Theorem 3.3.1: An abelian group algebra RG with $G = C_{m_0} \oplus C_{m_1}$ can be decomposed into a direct sum of minimally cyclic ideals as follows:

$$RG = <\theta_0(x_1)> \oplus <\theta_1(x_1)> \oplus \ldots \oplus <\theta_{t_1-1}(x_1)> \oplus$$
$$<\theta_0(x_0)R_0(x_1)> \oplus \ldots \oplus <\theta_{t_0-1}(x_0)R_0(x_1)> \oplus <R_0(x_0)R_0(x_1)>$$

Any cyclic ideal under MRX mapping can be obtained as a sum of a subset of minimally cyclic ideals in the above direct sum.

Proof: We first show that every ideal in the direct sum decomposition given above is

cyclic. For this purpose we need the following equation from [7].

$$z^k[a(x_0,x_1)] - x_0^k\, a(x_0,x_1) = \sum_{j=0}^{m_1-1} \sum_{i=0}^{k-1} \{ a_{<(j-1)_{m_1},(i-k)_{m_0}>} - a_{<j,(i-k)_{m_0}>} \} x_0^i x_1^j \tag{3.3.1}$$

For an ideal of the form $<\theta_i(x_1)>$, its structure implies that $\theta_{<j,0>} = a \in \mathbb{R}$; otherwise $\theta_{<j,i>} = 0$. For $k = 1$ to m_0-1, 'i' part of equation (3.3.1) is in the range 1 to m_0-1 and therefore we have $z^k[\theta_i(x_1)] = x_0^k\, \theta_i(x_1)$. Since $x_1^j\, \theta_i(x_1) = \theta_i'(x_1)$ has a similar structure to $\theta_i(x_1)$, it follows that $z^k[\theta_i'(x_1)] = x_0^k\, \theta_i'(x_1)$. Consequently $z^k[\theta_i(x_1)] = X^k\, \theta_i(x_1)$ for $k = 0$ to m_0m_1-1. Now any other element $a \in <\theta_i(x_1)>$ can be expressed as appropriate linear combinations of $X^k\, \theta_i(x_1)$, which are also $z^k[\theta_i(x_1)]$, for $k = 0$ to m_0m_1-1 and therefore $z(a) \in <\theta_i(x_1)>$.

For an ideal of the form $<\theta_i(x_0)R_0(x_1)>$, the structure of its idempotent a implies that $a_{<j,i>} = a_{<j',i>}$, $j \neq j'$. Therefore from equation (3.3.1) we have $z^k[\theta_j(x_0)\, R_0(x_1)] = x_0^k\, \theta_j(x_0)\, R_0(x_1)$, for $k = 1$ to m_0-1. Since $x_1^j\, \theta_j(x_0)\, R_0(x_1) = \theta_j(x_0)\, R_0(x_1)$ we have $z^k[\theta_i(x_1)] = X^k\, \theta_i(x_1)$ for $k = 0$ to m_0m_1-1. Now arguing as above it can be shown that $<\theta_j(x_0)\, R_0(x_1)>$ is cyclic.

In order to prove the theorem as explained in Remark 3.3.1 We have to show that each M_i is minimally cyclic. There are both ideals of form $<\theta_i(x_0)R_0(x_1)>$ and $<\theta_j(x_1)>$. Ideals of form $<\theta_i(x_0)R_0(x_1)>$ are easily seen to be minimal ideals of RG algebra. Therefore no nontrivial sub ideals can be considered in this case and consequently these ideals are minimally cyclic. Ideals of form $<\theta_j(x_1)>$, however, may contain a nontrivial sub ideal. Let M be any nontrivial sub ideal of $<\theta_j(x_1)>$ that is cyclic under MRX mapping. We now deduce that such an ideal M is nothing but $<\theta_j(x_1)>$ itself. Suppose $m(x_0,x_1)$ be the idempotent generator of M. Then from equation (3.3.1)

$$z(m(x_0,x_1)) - x_0 m(x_0,x_1) = m(x_1)$$

It will now be shown using the transform theory approach that $m(x_1)$ is 0. Since the nonzeros of the ideal $<\theta_j(x_1)>$ are of the form $(x_1^{\,j}+x_1^{\,m_1-j})R_0(x_0)$, the nonzeros of $m(x_1) \in$

$<\theta(x_1)>$ will also have this structure. On the other hand the nonzeros of M, a sub ideal of $<\theta(x_1)>$, does not have a similar structure and therefore at least one conjugate class from $(x_1^{\,j}+x_1^{\,m_1-j})R_0(x_0)$ is absent in the nonzeros of M. Let the zero positions of this conjugate class correspond to positions $x_0^{\,i}x_1^{\,j}$ and $x_0^{\,i'}x_1^{\,m_1-j}$. As $m(x_1) \in M$, by hypothesis, its spectrum should have zeros in these positions. Since the nonzeros of $m(x_1)$ are also of the form $(x_1^{\,j}+x_1^{\,m_1-j})R_0(x_0)$, the only way to satisfy these two conflicting conditions is to take $m(x_1) = 0$. This implies that $m_{<i,n-1>} = m_{<j,\ n-1>}$ for $i \neq j$. Repeating the above arguments for the case $z^j(m(x_0,x_1)) - x_0^{\,j}m(x_0,x_1)$, for j = 2 to m_1-1, it can be proved that $m_{<i,j>} = m_{<i',j>}$ for j= 1 to n−1 and $i \neq i'$.

If not all of $m_{<i,j>}$'s is zero, then $m(x_0,x_1) = m'(x_0)\,R_0(x_1) + m'(x_1)$. Since $m'(x_0)\,R_0(x_1)$ belongs to some ideal of form $<f(x_0)R_0(x_1)>$, which is orthogonal to $M \subset <\theta_j(x_1)>$, it follows that $m'(x_0)$ is zero. Since $<\theta_j(x_1)>$ corresponds to a minimal ideal of RC_{m_2} algebra, $m'(x_1)$ should be equal to $\theta_j(x_1)$. Consequently, M is equal to $<\theta_j(x_1)>$ and therefore ideal $<\theta_j(x_1)>$ is minimally cyclically closed under MRX mapping.□

The following theorem considers the case when G is direct product of r cyclic groups.

Theorem 3.3.2: An abelian group algebra RG with $G = C_{m_0} \oplus C_{m_1} \oplus ... \oplus C_{m_{r-1}}$ can be decomposed into a direct sum of minimally cyclic ideals as follows:

$$RG = I_0 \oplus I_1 \oplus ... \oplus I_{r-1}$$

where each group I_k can further be decomposed into

$$I_k = J_1^{(k)} \oplus J_2^{(k)} \oplus ... \oplus J_{t_k-1}^{(k)} \quad (k = 1 \text{ to } r-1)$$

and

$$I_0 = J_1^{(0)} \oplus J_2^{(0)} \oplus ... \oplus J_{t_0-1}^{(0)} \oplus \prod_{j=1}^{r-1} R_0(x_j)$$

where each $J_i^{(k)}$ is an ideal of the form $<\theta_i(x_k) \prod_{j=k+1}^{r-1} R_0(x_j)>$ and t_k^{th} ideal is $R_0(x_k)$ in the RC_{m_k} algebra. Any cyclic ideal under MRX mapping can be obtained as a sum of a subset

of minimally cyclic ideals in the above direct sum.

Proof: We prove the theorem by the method of induction on the number of cyclic groups 'r'. It is trivial for r = 1 and true for r = 2 from Theorem (3.3.1). Suppose it is true upto (r–1) . This implies that ideals that are minimally cyclic in group algebra RG' where $G' = C_{m_0} \oplus C_{m_1} \oplus ... \oplus C_{m_{r-2}}$ are:

1) Ideals of the form $<\theta(x_k) \prod_{j=k+1}^{r-1} R_0(x_j)>$, for k = 0 to r–3.

2) Ideals of the form $<\theta(x_{r-2})>$.

In the case of $G = G' \oplus C_{m_{r-1}}$ any ideal in the groups I_0 to I_{r-2} are of the form $<\theta(X')R_0(x_{r-1})>$, where $<\theta(X')>$ is a minimally cyclic ideal from RG'. For any $a(X) \in <\theta(X')R_0(x_{r-1})>$, since $x_{r-1}^j R_0(x_{r-1}) = R_0(x_{r-1})$, it follows that $a(X) = a(X')R_0(x_{r-1})$, where $a(X') \in <\theta(X')>$. In other words, the ideal $<\theta(X')R_0(x_{r-1})>$ of RG is the ideal $<\theta(X')>$ of RG' repeated m_{r-1} times. Because of this $(X'^j a(X')) = b(X')R_0(x_{r-1})$, where $X' \in G'$, which can also be written as $z^j[a(X')]R_0(x_{r-1})$ since $<\theta(X')>$ is cyclic. From this it follows that $<\theta(X')R_0(x_{r-1})>$ is cyclic. Since $<\theta(X')>$ is minimally cyclic ideal by induction assumption, it follows that the ideals in the group I_0 to I_{r-2} are minimally cyclic.

It remains to prove that the ideals of the form $<\theta(x_{r-1})>$ are minimally cyclic. First we shall show that this type of ideal is cyclic. Structure of the idempotent $\theta(x_{r-1})$ implies that $\theta_{<i_{r-1},0,...,0>} = a_k \in R$ $(0 \le i_{r-1} \le m_{r-1}-1)$; otherwise $\theta_i = 0$, where $< i_{r-1}, i_{r-2}, ..., i_0 >$ is the mixed–radix representation of ' i ' w.r.t. moduli $m_0, m_1, ..., m_{r-1}$. Because of this structure of the idempotent $\theta(x_{r-1})$ nonzero positions of $(X^i\theta(x_{r-1}))$ and and its corresponding cyclic shift $z^i[\theta(x_{r-1})]$ tally and therefore $(X^i\theta(x_{r-1})) = z^i[\theta(x_{r-1})]$ for i = 0 to n–1 ($|G| = n$). From this it can be shown that $<\theta(x_{r-1})>$ is cyclic as in Theorem 3.3.1.

We now show that the ideals of the form $<\theta(x_{r-1})>$ are minimally cyclic under MRX mapping. If possible $A \subset <\theta(x_{r-1})>$, any nontrivial sub ideal, be cyclic under MRX

mapping. Let $a(X)$ be the idempotent generator of ideal A. Our strategy is to show that $x_0^i a(X)$ is same as $z^i[a(X)]$ for $i = 1$ to m_0-1. This implies that $X^i a(X)$ is same as that of $z^i[a(X)]$ for $i = 0$ to $n-1$, where $n = \prod_{k=0}^{r-1} m_k$ is the order of G. From this the structure of $a(X)$ can be known which will be later on used to show that $<\theta_r(x_r)>$ is minimally cyclic. Now to prove first $z[a(X)] = x_0 a(X)$, we introduce auxiliary polynomials $a^0(X)$ to $a^{r-2}(X)$ derived from $a(X)$ as follows:

$$z[a(X)] - x_0 a(X) = a^0(x_1, x_2, \ldots, x_{r-1}) = a^0(X) \tag{3.3.2}$$

$$z^{M_1}[a^0(X)] - x_2 a^0(X) = a^1(x_2, \ldots, x_{r-1}) = a^1(X) \tag{3.3.3}$$

In general for $i = 0$ to $r-2$ we have

$$z^{M_i}[a^{i-1}(X)] - x_i a^{i-1}(X) = a^i(x_{i+1}, \ldots, x_{r-1}) = a^i(X) \tag{3.3.4}$$

Finally for $i = r-2$ we have

$$z^{M_{r-2}}[a^{r-3}(X)] - x_{r-2} a^{r-3}(X) = a^{r-2}(X) = a^{r-2}(x_{r-1}). \tag{3.3.5}$$

Note that in the above equations $a^i(x_{i+1}, \ldots, x_{r-1})$, for $i = 0$ to $r-2$, has nonzero locations only among the coordinates spanned by $G' = C_{m_{i+1}} \oplus C_{m_{i+2}} \oplus \ldots \oplus C_{m_{r-1}}$. Now we show that each of these polynomials $a^0(X)$ to $a^{r-2}(X)$ to be zero. Let us start from $a^{r-2}(X)$ which is of the form $a^{r-2}(x_{r-1})$. By a spectral argument similar to the one used in Theorem 3.3.1, $a^{r-2}(x_{r-1})$ is seen to be zero. Similarly when $b^{r-3}(X) = x_{r-2} a^{r-3}(X)$ is used in place of $a^{r-3}(X)$ in (3.3.4) , we get, say $b^{r-2}(x_{r-1})$, which again by spectral argument can be shown to be zero. Continuing in this fashion $m_{r-2}-1$ times, it can be deduced that $a^{r-3}(X)$ has the form $a(x_{r-2})R_0(x_{r-1})$. Since this belongs to an ideal of the form $<f(x_{r-2})R_0(x_{r-1})>$ which is orthogonal to $A \subset <\theta(x_{r-1})>$, it follows that $a^{r-3}(X)$ is equal to zero. This implies using (3.3.3), for $i = r-3$

$$x_{r-3} a^{r-4}(X) = z^{M_{r-3}}[a^{r-4}(X)]$$

Now we show $a^{r-4}(X)$ to be zero. As before using $b^{r-4}(X) = x_{r-3} a^{r-4}(X)$ in place of $a^{r-4}(X)$ in 3.3.3, for $i = r-3$, results in a polynomial $a'(X)$ that can be easily shown to

belong to the ideal of the form $<g(x_{r-2})R_0(x_{r-1})>$. Now invoking the orthogonality of this ideal with $A \subset <\theta(x_{r-1})>$, $a'(X)$ is seen to be zero. Therefore $x_{r-3}^2 a^{r-4}(X) = z^{2.M_{r-3}}[a^{r-4}(X)]$. In a similar way it can be proved that

$$x_{r-3}^k a^{r-4}(X) = z^{k.M_{r-3}}[a^{r-4}(X)], \text{ for } k = 1 \text{ to } m_{r-3}-1$$

This implies $a^{r-4}(X)$ to be cyclic among the coordinates $G'' = C_{m_{r-3}} \times C_{m_{r-2}} \times C_{m_{r-1}}$. Consequently $a^{r-4}(X)$ is of the form $a(x_{r-3})R_0(x_{r-2})R_0(x_{r-1})$ which belongs to an ideal orthogonal to ideal A. Therefore $a^{r-4}(X)$ is zero. Continuing in this way it can be shown successively that $a^{r-5}(X)$, and so on up to $a^0(X)$ are all zero polynomials , and finally we have $z[a(X)] = (x_0\ a(X))$. Arguing on similar lines it can be shown that $z^i[a(X)] = (x_0^i\ a(X))$ for $i = 1$ to m_0-1.

Now all these results imply that the structure of $a(X) = a(x_0) \prod_{k=1}^{r-1} R_0(x_k) + a(X')$, where $a(X')$ is among the coordinates spanned by $C_{m_1} \oplus \ldots \oplus C_{m_{r-1}}$. By orthogonal property again $a(x_0) = 0$. Since $a(X')$ is an idempotent in the group algebra consisting of direct product of r–1 cyclic groups, invoking the induction assumption the structure of the idempotent $a(X')$ is known. Again using the orthogonal property it can be seen that $a(X') = f(x_{r-1})$. Since $\theta(x_{r-1})$ is minimal it follows that $a(X) = f(x_{r-1}) = \theta(x_{r-1})$. Consequently $<\theta(x_{r-1})>$ is minimally cyclic under MRX mapping as was to be proved.□

The equivalent statement of Theorem 3.3.2 in the transform–domain is as follows. In a group algebra CG_r of group G, where $G = C_{m_0} \oplus C_{m_1} \oplus \ldots \oplus C_{m_{r-1}}$, over the complex space restricted to the real images, ideals generated by the following idempotent generators are cyclic:

1) An all zero and an all 1 vectors.

2) Idempotent generators of form $E^i(X_0)$; $i = 0$ to t_0-1

3) Idempotent generators of form $E^i(X_j) \prod_{k=0}^{j-1} R_0(X_k)$ for $i = 0$ to $t_j - 1$ and $j = 1$ to $r-1$.

4) Linear combinations of above.

For the following examples, we characterize cyclically closed abelian ideals in the transform domain , using the above theorem.

Example 3.3.1: $n = 8$; $m_0 = 4$, $m_1 = 2$

Table 3.3.1: List of Minimally cyclic Idempotents along with the General Form of Elements of Corresponding Minimally Cyclic Ideals in $C_4 \otimes C_2$ Group Algebra.

Srl no:	Minimally cyclic idempotent $E^i(X)$	Form of general element of ideal generated by $E^i(X)$
0	1	a'_0
1	$X_0+X_0^3$	$A_1X_0+A_1^*X_0^3$
2	X_0^2	$a'_2X_0^2$
3	$X_1(1+X_0+X_0^3+X_0^2)$	$a'_3X_1(a'_0+A_1X_0+A_1^*X_0^3+a'_2X_0^2)$

Example 3.3.2: $n = 32$; $m_0 = 4$, $m_1 = 8$

Table 3.3.2: List of Minimally cyclic Idempotents along with the General Form of Elements of Corresponding Minimally Cyclic Ideals in $C_4 \otimes C_8$ Group Algebra.

Srl no:	Minimally cyclic idempotent $E^i(X)$	Form of general element of ideal generated by $E^i(X)$
0	1	a'_0
1	$X_0+X_0^3$	$A_1X_0+A_1^*X_0^3$
2	X_0^2	$a'_2X_0^2$
3	$(X_1+X_1^7)\,(1+X_0+X_0^2+X_0^3)$	$(A_3X_1+A_3^*X_1^7)\,(a'_0+A_1X_0+a'_2X_0^2+A_1^*X_0^3)$
4	$(X_1^2+X_1^6)\,(1+X_0+X_0^2+X_0^3)$	$(A_4X_1^2+A_4^*X_1^6)\,(a'_0+A_1X_0+a'_2X_0^2+A_1^*X_0^3)$
5	$(X_1^3+X_1^5)\,(1+X_0+X_0^2+X_0^3)$	$(A_5X_1^3+A_5^*X_1^5)\,(a'_0+A_1X_0+a'_2X_0^2+A_1^*X_0^3)$
6	$X_1^4(1+X_0+X_0^2+X_0^3)$	$a'_6X_1^4(a'_0+A_1X_0+a'_2X_0^2+A_1^*X_0^3)$

Example 3.3.3: $n = 32$; $m_0 = 4$, $m_1 = 4$, $m_2 = 2$

Table 3.3.3: List of Minimally cyclic Idempotents along with the General Form of Elements of Corresponding Minimally Cyclic Ideals in $C_4 \otimes C_4 \otimes C_2$ Group Algebra

Srl no:	Minimally cyclic idempotent $E^i(X)$	Form of general element of ideal generated by $E^i(X)$
0	1	a'_0
1	$X_0+X_0^3$	$A_1X_0+A_1^*X_0^3$
2	X_0^2	$a'_2X_0^2$
3	$(X_1+X_1^3)\,(1+X_0+X_0^2+X_0^3)$	$(A_3X_1+A_3^*X_1^3)\,(a'_0+A_1X_0+a'_2X_0^2+A_1^*X_0^3)$
4	$X_1^2(1+X_0+X_0^2+X_0^3)$	$a'_4X_1^2(a'_0+A_1X_0+a'_2X_0^2+A_1^*X_0^3)$
5	$X_2(1+X_0+X_0^2+X_0^3)\,.$ $(1+X_1+X_1^2+X_1^3)$	$a'_5X_2(a'_0+A_1X_0+a'_2X_0^2+A_1^*X_0^3)\,.$ $(a'_6+A_7X_1+a'_8X_1^2+A_7^*X_1^3)$

3.4 Characterization of Cyclically Closed Abelian P–I Filters

In this section we take up the characterization of cyclically closed abelian filters. As explained earlier, any cyclically closed abelian filter necessarily belongs to some ideal that is closed under MRX mapping. We have characterized all such cyclically closed ideals in a given abelian group algebra. However, not all of the elements belonging to a cyclically closed abelian ideal can be taken as cyclically closed abelian P–I filters. This is due to the fact (Lemma 3.1.1) that any such cyclically closed abelian P–I filter, say s, should have additional property that the cyclic P–I matrix S_C and abelian P–I matrix S_A associated with it must be the same. Let us consider the following example by way of illustration.

Example 3.4.1: $n = 8$; $m_0 = 4$, $m_1 = 2$; $x_0^4 = x_1^2 = 1$.

Ideals closed under cyclic shifts are (1) $< \theta(x_1) >$ and (2) $< \theta(x_0)R_0(x_1) >$

Let us consider $s = (\ a_0 \ \ 0 \ \ 0 \ \ 0 \ \ a_1 \ \ 0 \ \ 0 \ \ 0\ ) \in < \theta(x_1) >$. In Section 3.1, we have seen that a P–I filter with this type of unit–sample response vector is a cyclically closed abelian P–I filter. Since $s \in < \theta(x_1) >$, all the abelian permutations of s also belong to $< \theta(x_1) >$. We now consider an abelian permutation of s, $x_0[s]$, and show that it is not a cyclically closed abelian P–I filter.

$$x_0[s] = (\ 0 \ \ a_1 \ \ 0 \ \ 0 \ \ 0 \ \ a_2 \ \ 0 \ \ 0 \ \ 0\ )$$

Using (3.1.1) and (3.1.2), we write system matrices S_C and S_A.

$$S_A = \begin{bmatrix} 0 & 0 & 0 & a_1 & 0 & 0 & 0 & a_2 \\ a_1 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & a_1 & 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_2 & 0 & 0 & 0 & a_1 \\ a_2 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 0 & a_1 & 0 \end{bmatrix}$$

$$S_C = \begin{bmatrix} 0 & 0 & 0 & a_2 & 0 & 0 & 0 & a_1 \\ a_1 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 \\ 0 & a_1 & 0 & 0 & 0 & a_2 & 0 & 0 \\ 0 & 0 & a_1 & 0 & 0 & 0 & a_2 & 0 \\ 0 & 0 & 0 & a_1 & 0 & 0 & 0 & a_2 \\ a_2 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 \\ 0 & a_2 & 0 & 0 & 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 & 0 & 0 & a_1 & 0 \end{bmatrix}$$

It can be seen that for $x_0[s]$, S_C and S_A are not equal. Therefore not all the elements of cyclically closed abelian ideals are cyclically closed abelian P–I filters. This is because the concept of a cyclically closed ideal is less restrictive than that of a cyclically closed abelian P–I filter.

Alternatively, such cyclic filters from a cyclically closed abelian ideal form a subspace which exhibits algebra isomorphism with the corresponding cyclic group algebra. We now characterize such a subspace of a cyclically closed abelian ideal. First we consider the case when $G = C_{m_0} \times C_{m_1}$ and later extend it to the case when G is direct product of r cyclic groups.

Theorem 3.4.1: In the group algebra RG with $G = C_{m_0} \times C_{m_1}$ the following elements are cyclically closed abelian filters (c.c.a.filters):

1) All elements of the ideals of the form $<\theta(x_0)R_0(x_1)>$.
2) All elements of the subspace consisting of elements of the form $f(x_1)$ belonging to an ideal of the type $<\theta(x_1)>$.
3) Elements obtained as linear combinations of the filters of above types.

: part 3 is trivial since MRX to cyclic mapping is a linear mapping on group algebra.

$<\theta(x_0)R_0(x_1)>$.

ging to the ideal is of the form $a(x_0,x_1) = a(x_0,x_1)\theta(x_0)R_0(x_1)$. Since

$x_1^j R_0(x_1)$ is same as that of $R_0(x_1)$, the above code word can be written as $a'(x_0)R_0(x_1)$, where $a'(x_0)$ is an element in the ideal $<\theta_0(x_0)>$ and using equation (3.3.1), we have $x_0^j a'(x_0)R_0(x_1) = z^j[a'(x_0)R_0(x_1)]$ for $j = 1$ to m_0-1. Therefore it follows that $X^k a'(x_0)R_0(x_1) = z^k[a'(x_0)R_0(x_1)]$ for $k = 0$ to m_0m_1-1. This shows that any element of the form $a'(x_0)R_0(x_1) \in < \theta_0(x_0)\, R_0(x_1) >$ qualify as c.c.a filters.

2) Ideals of the form $<\theta(x_1)>$.

Note that the this ideal is generated by the following set S consisting of polynomials $\{(x_0^i x_1^j\, \theta(x_1)\,;\, 0 \le i \le m_0-1\,,\, 0 \le j \le m_1-1\}$. Consider the polynomial $(x_0^i\, x_1^j\, \theta(x_1))$ where $i \neq 0$. Then we have

$$x_0^{m_0-i}\; x_0^i\, x_1^j\, \theta(x_1) = x_1^j\, \theta(x_1)$$

But

$$z^{m_0-i}[\,x_0^i\, x_1^j\, \theta(x_1)] = z^{m_0}[x_1^j \theta(x_1)] = x_1^{j+1}\theta(x_1)$$

It is clear that the above two expressions may not be equal in general. This happens whenever $i \neq 0$. And when $i = 0$, that is for elements of form $x_1^j\, \theta(x_1) = \theta'(x_1)$, according to (3.3.1) satisfy $X^k\theta'(x_1) = z^k[\theta'(x_1)]$, for $k = 0$ to m_0m_1-1, and hence qualify as c.c.a filters. By linearity property of MRX mapping all elements of the subspace generated by the set S' $= \{\, \theta(x_1),\, x_1\theta(x_1),\ldots,\, x_1^{m_1-1}\, \theta(x_1)\,\}$ also have the property that the corresponding abelian and cyclic shifts are equal and hence qualify as c.c.a filters. Such an element is of the form $f(x_1)$ as was to be proved.□

Remark 3.4.1: It should be noted that the subspaces spanned by ideals of type $<\theta(x_0)R_0(x_1)>$ and subspaces consisting of elements of the form $f(x_1) \in <\theta(x_1)>$ are isomorphic as a group algebra to the corresponding subspaces of the cyclic group algebra under MRX mapping. The structure of these subspaces indicate that the cyclic vector in the abelian case either a cyclic vector of C_{m_0} algebra repeated m_1 times or a cyclic vector of C_{m_1} algebra spread out in the abelian group algebra such that the corresponding

polynomial is of the form $f(x_1)$.

Now we take up the characterization of c.c.a filters for the case when $G = C_{m_0} \otimes C_{m_1} \otimes \ldots \otimes C_{m_{r-1}}$.

Theorem 3.4.2: In the group algebra RG with $G = C_{m_0} \otimes C_{m_1} \otimes \ldots \otimes C_{m_{r-1}}$ the following elements are cyclically closed abelian filters:

1) All elements of the ideals of the type $<\theta(x_0) \prod_{k=1}^{r-1} R_0(x_k)>$.

2) All elements of the subspace consisting elements of the form '$f(x_i) \prod_{k=i+1}^{r-1} R_0(x_k)$' belonging to an ideal of the type $<\theta(x_i) \prod_{k=i+1}^{r-1} R_0(x_k)>$ for $i = 1$ to $r-1$.

3) Elements obtained as linear combinations of the filters of above types.

Proof: We prove by induction on the number of cyclic groups in the direct product expansion of G. Theorem is trivial for $r = 1$ and true for $r = 2$ from Theorem 3.4.1. We suppose that the theorem is true for G consisting of $r-1$ cyclic groups. If $G' = C_{m_0} \otimes C_{m_1} \otimes \ldots \otimes C_{m_{r-2}}$ this implies that the following elements have c.c.a property in the RG' algebra:

Elements of the form '$f(x_i) \prod_{k=i+1}^{r-2} R_0(x_k)$' belonging to an ideal of the type

$$<\theta(x_i) \prod_{k=i+1}^{r-2} R_0(x_k)> \text{ for } i = 0 \text{ to } r-2.$$

If $G = G' \otimes C_{m_{r-1}}$ then as explained in Theorem 3.3.2 ideals are of the form $<\theta(x_i) \prod_{k=i+1}^{r-1} R_0(x_k)>$ and any element of the ideal is of the type $(a(X')\, R_0(x_{r-1}))$, where $a(X')$ is a c.c.a filter by induction assumption. From this it is simple to verify that $a(X')R_0(x_{r-1})$ is also a c.c.a filter. This accounts for all filter vectors mentioned in parts 1 and 2 except filters of the form $f(x_{r-1})$.

It remains to consider all possible filters that can be formed from the ideals of the

form $<\theta(x_{r-1})>$. It can be seen from the proof of Theorem 3.3.2 that all elements of the ideal of the type $(x_{r-1}^{j}\ \theta(x_{r-1}))$, for $j = 0$ to $m_{r-1}-1$, have the corresponding cyclic and abelian shifts equal and therefore all elements of the subspace generated by these elements qualify as c.c.a filters.

Now consider an element of the form $(X'^{a}\ \theta(x_{r-1}))$ where $X'^{a} \in G'$. Suppose 'i' be the first integer in the expansion of $X'^{a} = \prod_{i=0}^{r-2} x_i^{a_i}$ for which $a_i \neq 0$ and let $X'^{a} = x_i^{a_i} X'^{b}$. Then $x_i^{m_i-a_i}(X'^{a}\ \theta(x_{r-1})) = X'^{b}\ \theta(x_{r-1})$. On the other hand, $z^{M_i(m_i-a_i)}[X'^{a}\ \theta(x_{r-1})] = z^{M_i m_i}\ [X'^{b}\ \theta(x_{r-1})]$ which is not equal to $X'^{b}\ \theta(x_{r-1})$ in general. This can be seen as follows. If $i = r-2$, then $z^{M_i m_i}\ [X'^{b}\ \theta(x_{r-1})] = x_{r-2}\ \theta(x_{r-1})$ which may not be equal always. If $i \neq r-2$, then taking $b = 0$, it is easily seen that $z^{M_i m_i}\ [\theta(x_{r-1})] = \theta(x_{r-1})$ is not always true. Therefore all possible c.c.a filters from the ideal of the form $<\theta(x_{r-1})>$ are formed from the subspace consisting elements of the form $f(x_{r-1})$.□

Remark 3.4.2: As stated in Remark 3.4.1, in the general case also the subspaces spanned by the elements of the form $f(x_i) \prod_{k=i+1}^{r-1} R_0(x_k) \in <\theta(x_i) \prod_{k=i+1}^{r-1} R_0(x_k)>$ ($i = 0$ to $r-1$) are isomorphic as a group algebra to the corresponding subspaces of the cyclic group algebra under MRX mapping. The structure of these subspaces indicate that the cyclic vector in the abelian case either a cyclic vector of C_{m_0} algebra repeated $\prod_{k=0}^{r-1} m_k$ times or a cyclic vector of C_{m_i} algebra spread out in the abelian group algebra such that the corresponding polynomial is of the form $f(x_i) \prod_{k=i+1}^{r-1} R_0(x_k)$ for $i = 1$ to $r-1$.

3.5 Transform Domain Characterization of Cyclically Closed Abelian P-I Filters

Generalized Walsh Hadamard transform (GWHT) relates the Convolution operation

defined by the abelian permutation group, PG, in sample domain to point–wise multiplication in the transform domain. Whereas, Discrete fourier transform (DFT) relates the cyclic convolution operation in sample domain to point–wise multiplication in the transform domain. A system, T_c, that has the choice of being characterized by either of the convolutional relationships, can be described in the transform domain either by using DFT or by using GWHT. When the system, T_c, is used for manipulating the transform coefficients of input sequence, it can be viewed either as a cyclic filter (when DFT is used) or as an abelian filter (when GWHT is used). Figure 3.5.1 represents the transform domain interpretations of cyclically closed abelian P–I filter.

Theorem 3.4.2 gives all the elements in abelian group algebra whose cyclic permutations are same as abelian permutations. Its equivalent statement in the transform–domain is as follows.

In an abelian group algebra CG_r of group G where $G = C_{m_0} \otimes C_{m_1} \otimes \ldots \otimes C_{m_{r-1}}$, over the complex space restricted to the real images, the following are the cyclically closed abelian P–I filters:

1) All elements of the ideals of the type $< E^i(X_0) >$ for $i = 0$ to t_0-1.

2) All elements of the subspace consisting of elements of the form $< E^i(X_k) > \prod_{j=0}^{k-1} R_0(X_j)$ belonging to an ideal of the type $< E^i(X_k) \prod_{j=0}^{k-1} R_0(X_k) >$ for $i = 0$ to t_k-1 and $k = 1$ to $r-1$.

3) Elements obtained as linear combinations of filters of above types.

For the following examples, we list out all the cyclically closed abelian P–I filters in the transform domain.

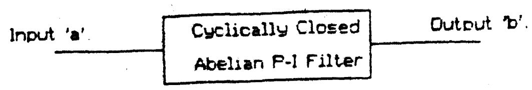

Fig. 3.5.1.a: Cyclically Closed Abelian P-I Filter

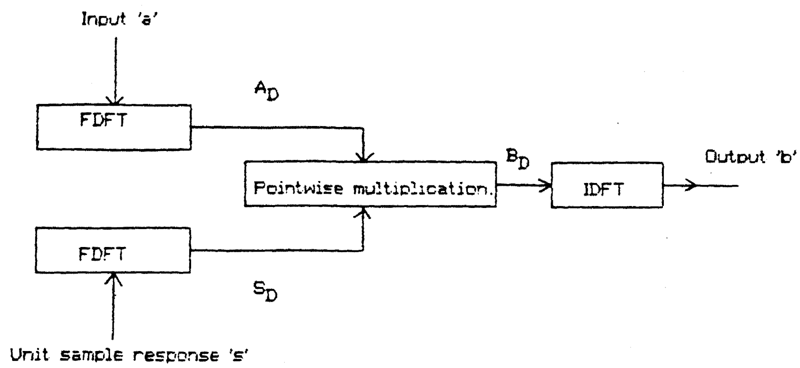

Fig. 3.5.1.b: Cyclically Closed Abelian P-I Filter as a Cyclic P-I Filter.

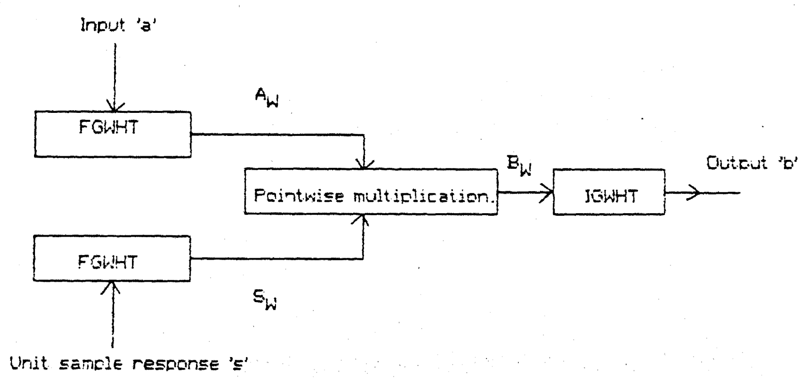

Fig. 3.5.1.c: Cyclically Closed Abelian P-I Filter as an Abelian P-I Filter

Fig 3.5.1 . Transform Domain Interpretations of Cyclically Closed Abelian P-I Filter

Example 3.5.1: $n = 8$; $r = 2$; $m_0 = 4$, $m_1 = 2$

Table 3.5.1: List of Minimally Cyclic Ideals and the General Form of Elements of these Ideals which qualify as Cyclically Closed Abelian P–I Filters in $C_4 \circledast C_2$ Group Algebra.

Srl no:	Minimally cyclically closed ideal $\langle E^i(X)\rangle$	Form of elements of $\langle E^i(X)\rangle$ whose cyclic and abelian permutations are identical
0	$\langle 1\rangle$	a'_0
1	$\langle X_0+X_0^3\rangle$	$A_1X_0+A_1^*X_0^3$
2	$\langle X_0^2\rangle$	$a'_2X_0^2$
3	$\langle X_1(1+X_0+X_0^2+X_0^3)\rangle$	$a'_3X_1(1+X_0+X_0^2+X_0^3)$

Example 3.5.2: $n = 32$; $r = 2$; $m_0 = 4$, $m_1 = 8$

Table 3.5.2: List of Minimally Cyclic Ideals and the General Form of Elements of these Ideals which qualify as Cyclically Closed Abelian P–I Filters in $C_4 \circledast C_8$ Group Algebra.

Srl no:	Minimally cyclically closed ideal $\langle E^i(X)\rangle$	Form of elements of $\langle E^i(X)\rangle$ whose cyclic and abelian permutations are identical
0	$\langle 1\rangle$	a'_0
1	$\langle X_0+X_0^3\rangle$	$A_1X_0+A_1^*X_0^3$
2	$\langle X_0^2\rangle$	$a'_2X_0^2$
3	$\langle (X_1+X_1^7)\,(1+X_0+X_0^2+X_0^3)\rangle$	$(A_3X_1+A_3^*X_1^7)\,(1+X_0+X_0^2+X_0^3)$
4	$\langle (X_1^2+X_1^6)\,(1+X_0+X_0^2+X_0^3)\rangle$	$(A_4X_1^2+A_4^*X_1^6)\,(1+X_0+X_0^2+X_0^3)$
5	$\langle (X_1^3+X_1^5)\,(1+X_0+X_0^2+X_0^3)\rangle$	$(A_5X_1^3+A_5^*X_1^5)\,(1+X_0+X_0^2+X_0^3)$
6	$\langle X_1^4(1+X_0+X_0^2+X_0^3)\rangle$	$a'_6X_1^4\,(1+X_0+X_0^2+X_0^3)$

where $p = \prod_{k=0}^{i-1} m_k$, A_{p+j} is a complex number and A^*_{p+j} is its complex conjugate .

Define $R_0(X_k) = \sum_{i=0}^{m_i-1} X_k^i$; $k = 0$ to $r-1$.

(4) Consider polynomials of following forms :

a) $S^j(X_0) = A_j X_0^j + A_j^* X_0^{m_0-j}$; $j = 0$ to $(t_0 - 1)$.

b) $S^j(X_i) \prod_{k=0}^{j-1} R_0(X_k)$; $j = 0$ to $(t_i - 1)$ and $i = 1$ to $(r-1)$.

(5) Let $S(X_0, X_1, .. ,X_{r-1})$ be the polynomial obtained by the linear combination of polynomials mentioned in Step 4 . Also, let $< i_0, i_1, ..., i_{r-1} >$ be the mixed–radix representation of ' i ' w.r.t. M, in the sample domain. Now, map the polynomial $S(X_0, X_1, .., X_{r-1})$ into a vector as $S = (S_0\ S_1\ S_2 \dots S_i \dots S_{n-1})$ where S_i is the coefficient of the term $\left(\prod_{\alpha=0}^{r-1} X_\alpha^{i_\alpha}\right)$ in $S(X_0, X_1, .., X_{r-1})$.

Each of these complex valued vectors, $S = (S_0\ S_1\ S_2 \dots S_i \dots S_{n-1})$, satisfying complex conjugacy relation can be taken as transform of the unit sample response vectors of cyclically closed abelian filters. Therefore, there are as many filters as the number of distinct vectors, S.

By way of illustration of the above procedure, we consider the following example.

Example 3.5.4:

(1) $n = 8$; $m_0 = 4$, $m_1 = 2$; $X_0^4 = X_1^2 = 1$

(2) conjugacy classes of m_0 and m_1 are (0) , (1 , 3) & (2) and (0) & (1) respectively. Number of conjugacy classes of m_0 and m_1 are $t_0 = 3$ and $t_1 = 2$ respectively.

(3) In polynomial form Conjugacy classes of m_0 and m_1 are

$S^0(X_0) = a'_0$, $S^1(X_0) = A_1X_0 + A_1^*X_0^3$ & $S^2(X_0) = a'_2\, X_0^2$ and $S^0(X_1) = a'_3$ & $S^1(X_1) = a'_4X_1$ respectively. (coefficients of single element conjugacy classes are reals)

Define $R_0(X_0) = 1 + X_0 + X_0^2 + X_0^3$.

(4) Consider polynomials of following forms:

(a) $S^j(X_0)$; $j = 0$ to 3.

(b) $S^j(X_1).R_0(X_0)$; $j = 0$ to 1.

(5) $S(X_0, X_1)$ be the polynomial obtained by the linear combination of polynomials mentioned in Step 4 .

$$S(X_0, X_1) = a'_0 + (\, A_1 X_0 + A_1^* X_0^3 \,) + (\, a'_2 X_0^2 \,) + a'_4 X_1 (\, 1 + X_0 + X_0^2 + X_0^3 \,) =$$

$$a'_0 + A_1X_0 + a'_2 X_0^2 + A_1^* X_0^3 + a'_4 X_1 + a'_4 X_1X_0 + a'_4 X_1X_0^2 + A_4 X_1X_0^3.$$

where A_i are complex coefficients.

This polynomial $S(X_0, X_1)$ is mapped into a vector as $S^n = (S_0\ S_1 \ldots S_i \ldots S_7)$ where S_i is the coefficient of the term $X_0^{i_0} X_1^{i_1}$; i_0 and i_1 mixed–radix representation of i in M in $S(X_0, X_1)$.

$$S = (a'_0\ \ A_1\ \ a'_2\ \ A_1^*\ \ a'_4\ \ a'_4\ \ a'_4\ \ a'_4)$$

Remark 3.5.1: In the sample domain by means of mixed–radix mapping we obtain the cyclic group algebra element for a given abelian group algebra element. But, in the transform domain this mapping is to be reversed for obtaining the corresponding cyclic group algebra element. This is illustrated in the example given below.

Example 3.5.5: Let $G = C_{m_0} \otimes C_{m_1} \otimes .. \otimes C_{m_{r-1}}$; $n = \prod_{\alpha=0}^{r-1}$; $X_\alpha^{m_\alpha} = 1$; $\alpha \in Z_r$ and let Z is a generator of a cyclic group, C_n, of order n. In the sample domain under MRX mapping $X_\alpha^{i_\alpha}$; $\alpha \in Z_r$ is mapped into $Z^{i_\alpha M_\alpha}$ where $M_\alpha = \prod_{k=0}^{\alpha-1} m_k$. Whereas, in the transform domain $X_\alpha^{i_\alpha}$ is mapped into $Z^{i_\alpha M_\alpha}$; $\alpha \in Z_r$ and $M_\alpha = \prod_{k=\alpha+1}^{r-1} m_k$.

In what follows, we give a procedure for finding the cyclic P–I filter for a given abelian P–I filter (Here, we are considering only those abelian filters that are closed under cyclic permutations).

Procedure for finding cyclic filter for a given abelian filter:

Let $S(X) = \sum_{i=0}^{n-1} S_i \left(\prod_{\alpha=0}^{r-1} X_\alpha^{i_\alpha} \right)$ be a cyclically closed abelian filter whose equivalent cyclic filter vector is to be found.

By applying the MRX mapping, $X_\alpha^{i_\alpha} = Z^{i_\alpha M_\alpha}$; $\alpha \in Z_r$ and $M_\alpha = \prod_{k=\alpha+1}^{r-1} m_k$, to $S(X)$, corresponding cyclic filter $S(Z)$ can be obtained. Then take IDFT of $S(Z)$, or take IGWHT of $S(X)$. Denote it by $s(X)$. In the sample–domain, by substituting $Z^{i_\alpha M_\alpha}$ for $X_\alpha^{i_\alpha}$; $\alpha \in Z_r$ and $M_\alpha = \prod_{k=0}^{\alpha-1} m_k$ we can obtain $s(Z)$, where $s(Z)$ is IDFT of $S(Z)$.

Example 3.5.6: $n = 8$; $m_0 = 4$, $m_1 = 2$; $X_0^4 = X_1^2 = Z^8 = 1$

Let $S(X) = X_0 + X_0^3$ be the transform of a unit–sample response element of a cyclically closed abelian P–I filter whose equivalent cyclic P–I filter vector is to be found.

By applying transform domain MRX mapping, $X_0 = Z^2$ and $X_1 = Z$, to $S(X)$ we get the corresponding cyclic P–I filter vector $S(Z)$ as

$$S(Z) = Z^2 + Z^6.$$

Its sample domain vector, $s(Z)$, is obtained by taking IDFT of $S(Z)$.

$$s(Z) = 1 - Z^2 + Z^4 - Z^6$$

The other way to find $s(Z)$ is, first find the sample domain vector, $s(X)$, of $S(X)$ (by taking IGWHT of $S(X)$).

$$s(X) = 1 - X_0^2 + X_1 + X_1 X_0^2.$$

Applying sample domain MRX mapping, $X_0 = Z$ and $X_1 = Z^4$, to $s(X)$ we get the corresponding cyclic P–I filter vector $s(Z) = 1 - Z^2 + Z^4 - Z^6$.

3.6 A Procedure for Identifying 2–D Cyclically Closed Abelian P–I Filters

In this section, we proceed with the filtering of 2–D finite length discrete data on the lines of the facts discussed so far. To begin with, we review some of the concepts

related to 2–D P–I systems [9].

Here, the signal space of our interest is the space consisting of all n x m matrices over reals. It can be treated as a vector space, V, of real n x m matrices. A 2–D finite discrete system is a mapping of the type V ——> V. Let us take, for instance, an arbitrary input signal a from V, and permute the rows of ' a ' by members of a transitive abelian permutation group PG_1 of order m and the columns by another transitive abelian permutation group PG_2 of order n. If the rows and columns of the corresponding output signal of the system T also gets permuted exactly in the same manner, then T is invariant to such permutations. Then we say that T is a 2–D permutation–invariant (2–D P–I) system relative to the pair of transitive abelian permutation groups PG_1 and PG_2.

Thus, with a vector space of matrices over reals treated as a signal space, the operation of a generalized convolution admits of being treated as a 2 D P–I system relative to PG_1 and PG_2. In other words a 2–D P–I system is described by:

$$b_{k,l} = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} s_{k \ominus i \, , \, l \nabla j} \, a_{i,j} \quad ; k \in Z_n \, , l \in Z_m \tag{3.6.1}$$

where a, b and s are respectively the input , output and systems unit sample response arrays ; ∇ and $\ominus$ respectively denote pointwise subtraction operation in two mixed–radix number systems, one with mixed–radix set $M = \{ m_0, m_1, \dots , m_{r-1} \}$ w.r.t which PG_1 is defined and the other with mixed–radix set $N = \{ n_0, n_1, \dots , n_{s-1} \}$ w.r.t which PG_2 is defined.

Definition 3.6.1: A 2–D P–I filter is a system T which maps the space of real n x m matrices, V, into a linear subspace of V in which the rows and columns of the matrices are invariant relative to PG_1 and PG_2 respectively and its input–output relationship is characterized by (3.6.1).

Generally, we call this filter as a 2–D abelian P–I filter. If PG_1 and PG_2 are cyclic permutation groups, then, we call it as a 2–D cyclic P–I filter. If the effect of permutations

belonging to cyclic permutation groups PC_1 (transitive cyclic permutation group of degree m) and PC_2 (transitive cyclic permutation group of degree n) is same as that of permutations belonging to the permutation groups PG_1 and PG_2 on rows and columns of the unit–sample response matrix respectively, then the filter being characterized by this type of unit–sample response matrix is called a 2–D cyclically closed abelian P–I filter .

Definition 3.6.2: A *2–D cyclically closed abelian P–I filter* is a system T which maps the space of real n x m matrices, V, into a linear subspace of V in which the rows and columns of the matrices are invariant relative to the pair of permutation groups PC_1 and PG_1 and PC_2 and PG_2 respectively and its input–output relationship can be characterized either by (3.6.1) or by the following equation.

$$b_{k,l} = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} s_{(k-i)\bmod n \;,\; (l-j)\bmod m}\; a_{i,j} \quad ; k \in Z_n \;, l \in Z_m \tag{3.6.2}$$

This is a 2–D cyclic convolutional relationship which is a special case of the generalized convolution defined as above in (3.6.1).

Alternatively, 2–D P–I system can be described in the transform domain using 2–D GWHT as follows.

A 2–D GWHT defined with respect to the mixed–radix sets $M = \{ m_0, m_1, .., m_{r-1}\}$, $N = \{n_0, n_1, .., n_{s-1}\}$ maps $a \in V$.into a matrix of complex numbers, A, in the transform domain.These two matrices are related by the following equations.

$$A_{k,l} = \sum_{i=0}^{n-1} \sum_{j=0}^{m-1} \Big(\prod_{\alpha=0}^{r-1} \gamma_{n_\alpha}^{-i_\alpha k_\alpha} \Big) \Big(\prod_{\alpha=0}^{s-1} \gamma_{m_\alpha}^{-j_\alpha l_\alpha} \Big)\, a_{i,j} \;; k \in Z_n \;, l \in Z_m \tag{3.6.3}$$

where γ_{m_α} and γ_{n_α} are m_α^{th} and n_α^{th} roots of unity respectively.

$$a_{i,j} = (1/mn) \sum_{k=0}^{n-1} \sum_{l=0}^{m-1} \left(\prod_{\alpha=0}^{r-1} \gamma_{n_\alpha}^{\ i_\alpha k_\alpha} \right) \left(\prod_{\alpha=0}^{s-1} \gamma_{m_\alpha}^{\ j_\alpha l_\alpha} \right) A_{k,l} \ ; \ i \in Z_n , j \in Z_m \tag{3.6.4}$$

Equations (3.6.3) and (3.6.4) are known as 2–D Generalized Walsh–Hadamard transform pair.

If the matrix a belongs to V, then the samples of the transform matrix A satisfy the following relationship,generally known as *complex–conjugacy relation* between the transform coefficients.

$$A_{i,j} = A^{*}_{n_\theta i, m \nabla j} \tag{3.6.5}$$

where * denotes complex–conjugate, Θ and ∇ respectively denote pointwise subtraction operation in two mixed–radix number systems , one with mixed–radix set M = { m_0, m_1, ... , m_{r-1} } and the other with mixed–radix set N = { n_0, n_1, ... , n_{s-1} }

2–D GWHT defined as in (3.6.3) and (3.6.4) replaces the convolution operation defined as in (3.6.1) with pointwise multiplication. Therefore, in the transform domain (3.6.1) can be expressed as

$$B_{i,j} = S_{i,j} \cdot A_{i,j} \tag{3.6.6}$$

where $B \underset{\text{FGWHT}}{\overset{\text{IGWHT}}{\rightleftarrows}} b$, $S \underset{\text{FGWHT}}{\overset{\text{IGWHT}}{\rightleftarrows}} s$ and $A \underset{\text{FGWHT}}{\overset{\text{IGWHT}}{\rightleftarrows}} a$

Now with this background and by making use of the procedure given in Section 3.5 for identification of cyclically closed abelian P–I filters, we will give a procedure for identification of 2–D cyclically closed abelian P–I filters.

Procedure for identifying 2–D cyclically closed abelian P–I filters:

1) For a given pair of ' n , m ' choose mixed–radix sets N = { n_0, n_1, ... , n_i , ... , n_{r-1} } and M = { m_0, m_1, ... , m_i , ... , m_{s-1} }.

2) Find out the conjugacy classes for all m_i , $0 \le i \le r-1$. Let t_i be the number of

conjugacy classes of an m_i .

3) Express each such conjugacy class as a polynomial in X_i . Denote the j^{th} conjugacy class of m_i by $S^j(X_i)$.

where $S^j(X_i) = A_{p+j} X_i^{j} + A^{*}_{p+j} X_i^{m_0 - j}$; $j = 0$ to (t_i-1) , $i = 0$ to $r-1$, $p = \prod_{k=0}^{i-1} m_k$

A_{p+j} is a complex number and A^{*}_{p+j} is its complex conjugate.

Define $R_0(X_k) = \sum_{i=0}^{m_i - 1} X_k^i$; $k = 0$ to $r-1$.

4) Consider polynomials of following forms:

(a) $S^j(X_0) = A_j X_0^{j} + A_j^{*} X_0^{m_0 - j}$; $j = 0$ to (t_0-1).

(b) $S^j(X_i) . \prod_{k=0}^{j-1} R_0(X_k)$; $j = 0$ to (t_i-1) and $i = 1$ to $(r-1)$.

5) Let $S(X_0, X_1, .., X_{r-1})$ be the polynomial obtained by the linear combination of polynomials mentioned in step 4 . Also , let $< i_0, i_1, ..., i_{r-1} >$ be the mixed–radix representation of ' i ' w.r.t. M . Now , map the polynomial $S(X_0, X_1, .., X_{r-1})$ into a vector as $S = (S_0\ S_1\ S_2 \dots S_i \dots S_{n-1})$ where S_i is the coefficient of the term $(\prod_{\alpha=0}^{r-1} X_\alpha^{i_\alpha})$ in $S(X_0, X_1, .., X_{r-1})$.

6) Repeat from Step 2 to Step 5 for the second mixed–radix set. Let us denote the vectors corresponding to the two mixed–radix sets M and N by $S^m = (S_0^m\ S_1^m\ S_2^m \dots S_j^m \dots S_{m-1}^m)$ and $S^n = (S_0^n\ S_1^n\ S_2^n \dots S_i^n \dots S_{n-1}^n)$ respectively.

7) Construct a matrix S using the following equation .

$$S_{i,j} = S_i^n . S_j^m \quad ; i \in Z_n \text{ and } j \in Z_m \tag{3.6.7}$$

This matrix S is the transform of the unit sample response of a 2–D cyclically closed abelian P–I filter. For obtaining the corresponding cyclic P–I filter, first apply the mapping to the vectors S^n and S^m as mentioned in Example 3.5.5 and then construct the matrix as given in (3.6.7)

Example 3.6.1:

1) $n = 8$, $m = 8$, $N = \{ n_0 = 4 , n_1 = 2 \}$ and $M = \{ m_0 = 4 , m_1 = 2 \}$

First consider n , N .

2) From Example 3.5.4, we obtain $S^n = (a'_0 \ A_1 \ a'_2 \ A^*_1 \ a'_4 \ a'_4 \ a'_4 \ a'_4)$ and $S^m = (b'_0 \ B_1 \ b'_2 \ B^*_1 \ b'_4 \ b'_4 \ b'_4 \ b'_4)$

3) Construct a matrix S using the following equation.

$S_{i,j} = S^n_i \cdot S^m_j \quad ; i \in Z_8$ and $j \in Z_8$

This matrix S is the transform of the unit sample response of a 2–D cyclically closed abelian P–I filter.

The following is the general form of transform of the unit sample response matrix of a 2–D cyclically closed abelian P–I filter of dimension 8.

$$S = \begin{bmatrix} c'_0 & D_0 & a'_0 & D^*_0 & b'_0 & b'_0 & b'_0 & b'_0 \\ C_1 & D_1 & A_1 & D^*_1 & B_1 & B_1 & B_1 & B_1 \\ c'_2 & D_2 & a'_2 & D^*_2 & b'_2 & b'_2 & b'_2 & b'_2 \\ C^*_1 & D^*_1 & A^*_1 & D_1 & B^*_1 & B^*_1 & B^*_1 & B^*_1 \\ c'_3 & D_3 & a'_3 & D^*_3 & b'_3 & b'_3 & b'_3 & b'_3 \\ c'_3 & D_3 & a'_3 & D^*_3 & b'_3 & b'_3 & b'_3 & b'_3 \\ c'_3 & D_3 & a'_3 & D^*_3 & b'_3 & b'_3 & b'_3 & b'_3 \\ c'_3 & D_3 & a'_3 & D^*_3 & b'_3 & b'_3 & b'_3 & b'_3 \end{bmatrix}$$

3.7 Equivalent Cyclic P–I Filters based on CRT Mapping

In the above discussion, we have shown that under MRX mapping only a part of abelian group algebra, namely vector space formed by cyclically closed abelian P–I filters, is isomorphic to a part of cyclic group algebra and that any cyclic P–I filter with unit sample response element from this part of cyclic group algebra has the structures of both

cyclic and abelian P–I filters inherently. In this section we undertake a study of cyclic mappings based on another important type of mapping namely CRT. This type of mapping is possible only when the orders of component cyclic sub groups are pairwise relatively prime. In this case it will be shown that algebras are isomorphic. As a consequence of this a cyclic P–I filter can be realized as an equivalent abelian P–I filter. This type of realization has advantages compared to the realization based on G–T algorithm as will be shown later.

We begin by showing that under CRT mapping algebras are isomorphic.

Theorem 3.7.1: Let $n = \prod_{\alpha=0}^{r-1} m_\alpha$ be a product of integers relatively prime in pairs. Group algebra RG of a group G where $G = C_{m_0} \otimes C_{m_1} \otimes \ldots \otimes C_{m_{r-1}}$ is isomorphic to the group algebra RC of a cyclic group of order n under CRT mapping.

To prove this theorem we need the following two lemmas.

Lemma 3.7.1: Under CRT mapping, if a(X) and b(X) of RG are mapped into a(Z) and b(Z) of RC then a(X) + b(X) is mapped into a(Z) + b(Z).

Proof: Any a(X) and b(X) of RG can be written as $a(X) = \sum_{i=0}^{n-1} a_i (\prod_{\alpha=0}^{r-1} X_\alpha^{i_\alpha})$ and $b(X) = \sum_{i=0}^{n-1} b_i (\prod_{\alpha=0}^{r-1} X_\alpha^{i_\alpha})$. Under CRT mapping,

$$a(X) \xrightarrow{\text{CRT}} a(Z) = \sum_{I=0}^{n-1} a_I Z^I \text{ and } b(X) \xrightarrow{\text{CRT}} b(Z) = \sum_{I=0}^{n-1} b_I Z^I$$

where $I = \sum_{\alpha=0}^{r-1} i_\alpha M_\alpha N_\alpha \pmod{n}$.

Consider the CRT mapping of [a(X) + b(X)],

$$[\sum_{i=0}^{n-1} a_i (\prod_{\alpha=0}^{r-1} X_\alpha^{i_\alpha}) + \sum_{i=0}^{n-1} b_i (\prod_{\alpha=0}^{r-1} X_\alpha^{i_\alpha})] = \sum_{i=0}^{n-1} (a_i + b_i) (\prod_{\alpha=0}^{r-1} X_\alpha^{i_\alpha})$$

$$\xrightarrow{\text{CRT}} \sum_{I=0}^{n-1} (a_I + b_I) Z^I \qquad (3.7.1)$$

Now considering the sum of CRT mapping of a(X) and b(X), we have

$$a(X) + b(X) \xrightarrow{CRT} \left(\sum_{I=0}^{n-1} a_I Z^I\right) + \left(\sum_{I=0}^{n-1} b_I Z^I\right) = \sum_{I=0}^{n-1} (a_I + b_I) Z^I \qquad (3.7.2)$$

From (3.7.1) and (3.7.2), it can be seen that CRT mapping of [a(X) + b(X)] is same as that of a(X) and b(X).□

Lemma 3.7.2: Let $G = C_{m_0} \otimes C_{m_1} \otimes \ldots \otimes C_{m_{r-1}}$; let X_α, $\alpha \in Z_r$, be a generator of a cyclic sub group C_{m_α} of order m_α, let Z be the generator of a cyclic group C_n of order n where $n = \prod_{\alpha=0}^{r-1} m_\alpha$. Under a mapping based on CRT G and C_n are isomorphic.

Proof: Under a mapping based on CRT, $i_\alpha = I \bmod m_\alpha$ and $I = \sum_{\alpha=0}^{r-1} i_\alpha M_\alpha N_\alpha \pmod n$ where $M_\alpha = n/m_\alpha$ and N_α is the solution of $N_\alpha M_\alpha + n_\alpha m_\alpha = 1$. Then $\prod_{\alpha=0}^{r-1} X_\alpha^{i_\alpha}$ and $\prod_{\alpha=0}^{r-1} X_\alpha^{j_\alpha}$ are mapped into Z^I and Z^J respectively. Consider the product of CRT mappings of Z^I and Z^J,

$$Z^I \xrightarrow{CRT} \prod_{\alpha=0}^{r-1} X_\alpha^{i_\alpha} \text{ and } Z^J \xrightarrow{CRT} \prod_{\alpha=0}^{r-1} X_\alpha^{j_\alpha}$$

$$(Z^I.Z^J) \xrightarrow{CRT} \left(\prod_{\alpha=0}^{r-1} X_\alpha^{i_\alpha} . \prod_{\alpha=0}^{r-1} X_\alpha^{j_\alpha}\right) = \left(\prod_{\alpha=0}^{r-1} X_\alpha^{(i_\alpha + j_\alpha) \bmod m_\alpha}\right) \qquad (3.7.3)$$

Now considering the CRT mapping of Z^{I+J}, we have

$$Z^{I+J} \xrightarrow{CRT} = \left(\prod_{\alpha=0}^{r-1} X_\alpha^{(i_\alpha + j_\alpha) \bmod m_\alpha}\right) \qquad (3.7.4)$$

Since (3.7.3) and (3.7.4) are equal G and C_n are isomorphic under CRT mapping.□

Proof: In order to prove that RG and RC are isomorphic, first we prove that [a(X) + b(X)] is same as that of [a(Z) + b(Z)] under CRT mapping, where $a(X) \xrightarrow{CRT} a(Z)$ and $b(X) \xrightarrow{CRT} b(Z)$. This is already proved in Lemma 3.7.1. Next, we prove that the product

of any a(X) and b(X) of RG under CRT mapping, is the same as that of a(Z) and b(Z) of RC_n. Since an element of RG is a linear combination of the elements of G and linearity property holds for CRT mapping, it is enough to show that the CRT mapping of the product of any two elements of C_n is the same as that of the product of the CRT mapping of the individual elements of C_n. This is proved in Lemma 3.7.2.□

Corollary 3.7.1: Under CRT mapping, product of a(Z) b(Z) mod $(Z^n - 1)$ in RC is the same as that of a(X) b(X) mod $(X_0^{m_0} - 1), (X_1^{m_1} - 1), ., (X_{r-1}^{m_{r-1}} - 1)$ in RG where a(Z) $\xrightarrow{CRT}$ a(X) and b(Z) $\xrightarrow{CRT}$ b(X).

Proof: Its proof is contained in the proof of Theorem 3.7.1.□

Corollary 3.7.2: Under CRT mapping, an ideal of RC mapped into an ideal of RG.

Proof: This a consequence of the above corollary and the Lemma 3.7.1.□

Remark 3.7.1: Theorem 3.7.1 has an interesting consequence that any cyclic P–I filter whose dimension is a product of pairwise relatively prime integers, has an equivalent realization as an abelian P–I filter. As proved in the theorem relevant mapping is given by the CRT. It will be shown later that the realization of abelian P–I filter has some advantages compared to the realization of the corresponding cyclic P–I filter using Good–Thomas algorithm in respect of address shuffling.

We illustrate above concepts with the following example.

Example 3.7.1: $n = 6$; $m_0 = 2$, $m_1 = 3$.

CRT mapping : $i = 4\,i_1 + 3\,i_0 \mod 6$ where $i \in Z_6$, $i_1 \in Z_3$ and $i_0 \in Z_2$.

Consider the following cyclic P–I filter.

$$\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} = \begin{bmatrix} s_0 & s_5 & s_4 & s_3 & s_2 & s_1 \\ s_1 & s_0 & s_5 & s_4 & s_3 & s_2 \\ s_2 & s_1 & s_0 & s_5 & s_4 & s_3 \\ s_3 & s_2 & s_1 & s_0 & s_5 & s_4 \\ s_4 & s_3 & s_2 & s_1 & s_0 & s_5 \\ s_5 & s_4 & s_3 & s_2 & s_1 & s_0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix}$$

We know that the above matrix computation is equivalent to the polynomial multiplication a(Z)s(Z) mod $(Z^n - 1)$. By Theorem 3.7.1 this is equivalent to the polynomial multiplication a(X)s(X) mod $(X_0^{m_0} - 1)$, $(X_1^{m_1} - 1)$, ., $(X_{r-1}^{m_{r-1}} - 1)$. The corresponding mapping is given by CRT mapping. The corresponding matrix equivalent is given by

$$\begin{bmatrix} b_0 \\ b_3 \\ b_4 \\ b_1 \\ b_2 \\ b_5 \end{bmatrix} = \begin{bmatrix} s_0 & s_3 & s_2 & s_5 & s_4 & s_1 \\ s_3 & s_0 & s_5 & s_2 & s_1 & s_4 \\ s_4 & s_1 & s_0 & s_3 & s_2 & s_5 \\ s_1 & s_4 & s_3 & s_0 & s_5 & s_2 \\ s_2 & s_5 & s_4 & s_1 & s_0 & s_3 \\ s_5 & s_2 & s_1 & s_4 & s_3 & s_0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_3 \\ a_4 \\ a_1 \\ a_2 \\ a_5 \end{bmatrix}$$

Corresponding cyclic P–I filter output is obtained by applying inverse CRT mapping to the output of abelian P–I filter.

Since the extension field required for defining both the DFT and GWHT are same, it is possible to consider the spectral equivalent of the corresponding sample domain CRT mapping. In the following theorem we show that the CRT mapping considered in the sample domain also maps a GWHT coefficient to a corresponding DFT coefficient.

Theorem 3.7.2: Let 't' be the CRT mapping; let A^D and A^G denote the DFT and GWHT of *a*. Then A_j^G is equal to $A_{t(j)}^D$.

Proof: GWHT defined with respect to $\{m_0, m_1, \ldots, m_{r-1}\}$ maps a vector *a* as

$$A^G_{<j_{r-1}\, j_{r-2}\, j_1\, j_0>} = \sum_{i=0}^{n-1} a_{<i_{r-1}\, i_{r-2}\, i_1\, i_0>} [\, \gamma_{m_{r-1}}^{i_{r-1}j_{r-1}}\, \gamma_{m_{r-2}}^{i_{r-2}j_{r-2}} \cdots \gamma_{m_1}^{i_1 j_1}\, \gamma_{m_0}^{i_0 j_0} \,]$$

(3.7.5)

where $<i_{r-1}\, i_{r-2}\quad i_1\, i_0>$ and $<j_{r-1}\, j_{r-2}\quad j_1\, j_0>$ are the mixed–radix representations of i

and j.

Under CRT mapping, i and j are mapped into t(i) and t(j) where $t(i) = [\sum_{\alpha=0}^{r-1} N_\alpha M_\alpha i_\alpha] \bmod n$ and $t(j) = [\sum_{\alpha=0}^{r-1} N_\alpha M_\alpha j_\alpha] \bmod n$ and $t(i) \bmod m_\alpha = i_\alpha$; $\alpha \in Z_r$, and also $t(i)t(j) = t(ij) \bmod n$.

Applying CRT mapping to (3.7.5), we get,

$$\sum_{i=0}^{n-1} a_{t(i)} \gamma_n^{t(ij)} = \sum_{i=0}^{n-1} a_{t(i)} \gamma_n^{t(i)t(j)} \tag{3.7.6}$$

which is same as $A^D_{t(j)}$.

Therefore $t(j)^{th}$ DFT coefficient of 'a' is same as the j^{th} GWHT coefficient □

Remark 3.7.2: This is unlike the case of Good–Thomas (G–T) algorithm which requires two different address shuffling, one for the spectral domain and the other for the sample domain. Therefore there is an implementation advantage in using GWHT compared to G–T algorithm in realizing a cyclic P–I filter in that we can do with only one type of address shuffling both in the spectral domain and sample domain, namely CRT mapping.

The block diagrams for realizing cyclic P–I filter using distinct approaches of GWHT and G–T algorithm are given in Figure 3.7.1.

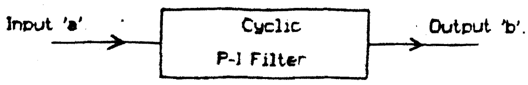

Fig. 3.7.1.a: Cyclic P-I Filter

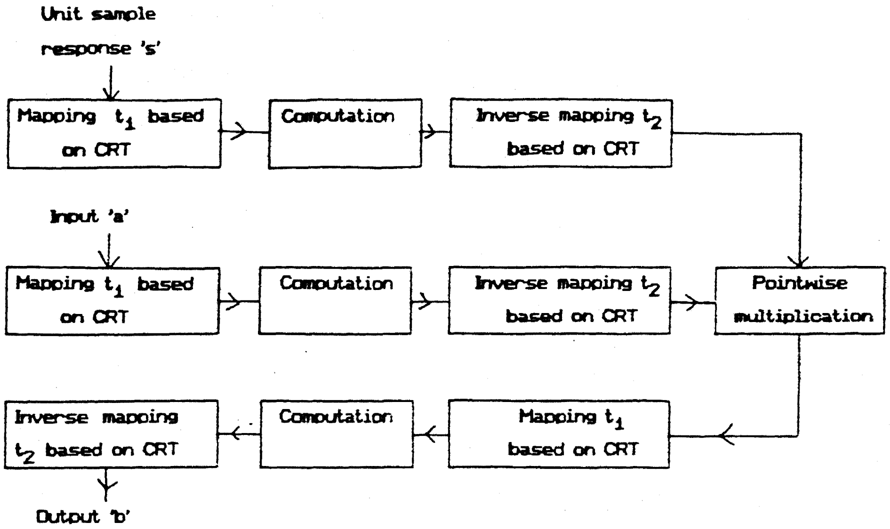

Fig. 3.7.1.b: Implementation of a Cyclic P-I Filter using G-T FFT based on CRT

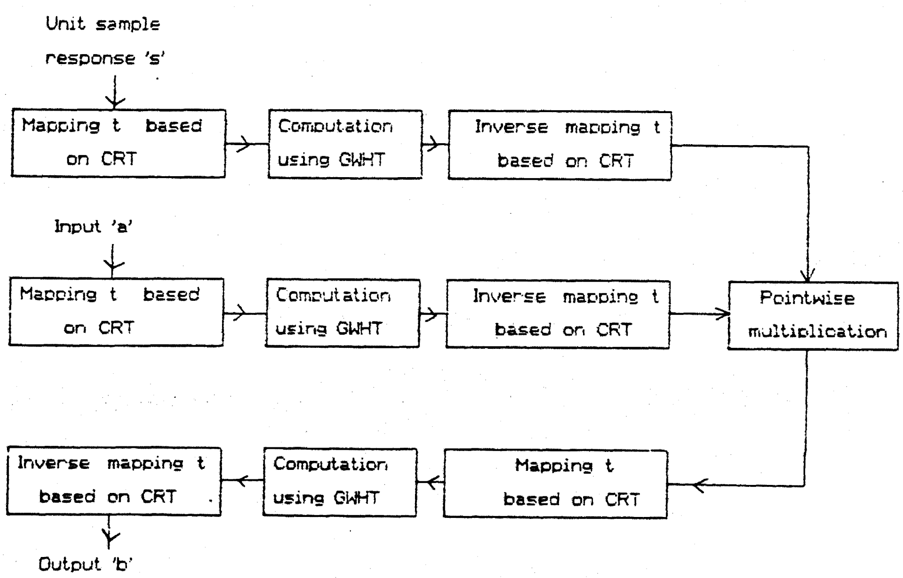

Fig. 3.7.1.c: Implementation of Cyclic P-I Filter using GWHT based on CRT Mapping

Fig 3.7.1 : Implementation of a Cyclic P-I Filter using G-T FFT Algorithm and using GWHT based on CRT

CHAPTER 4
CONCLUSIONS

In this thesis, we have considered group algebraic approach to the study of P–I filters. In this approach, a P–I filter is seen as a mapping that maps any element of a group algebra into an element of an ideal in that group algebra. The class of P–I filters that maps elements from a group algebra into an ideal is itself shown to be an ideal. This facilitates classification of P–I filters based on ideals in group algebra. Specifically, ideal theoretic results have been used to characterize P–I filters that have got the structures of both cyclic and abelian P–I filters inherently. We call this type of filter as cyclically closed abelian P–I filter. It is shown that under MRX mapping, only a part of abelian group algebra, namely subspace spanned by the class of cyclically closed abelian P–I filters, is isomorphic to the corresponding part of cyclic group algebra. Next, we have considered CRT mapping (When the orders of component cyclic groups are pairwise relatively prime) and shown that under this mapping, a cyclic group algebra is isomorphic to an abelian group algebra. Then we have shown that all cyclic P–I filters whose dimensions are products of pairwise relatively prime numbers have equivalent abelian P–I filter realizations.

4.1 Summary of the Results

The results obtained in this thesis are summarized below.

1) A class of P–I filters that maps elements from a group algebra into an ideal is itself shown to be an ideal. Consequently, abelian P–I filters are classified based on the ideals in abelian group algebra and cyclic P–I filters are classified based on the ideals in cyclic group algebra.

2) A P–I filter that is invariant relative to both cyclic and abelian permutations is

called a cyclically closed abelian P–I filter. This type of P–I filters are interpreted as elements of cyclically closed ideals in abelian group algebras and a complete characterization of these filters is obtained. In the group algebra RG with $G = C_{m_0} \otimes C_{m_1} \otimes \ldots \otimes C_{m_{r-1}}$ the following elements are cyclically closed abelian filters:

a) All elements of the ideals of the type $<\theta(x_0) \prod_{k=1}^{r-1} R_0(x_k)>$.

b) All elements of the subspace consisting of elements of the form $'f(x_i) \prod_{k=i+1}^{r-1} R_0(x_k)'$ belonging to an ideal of the type $<\theta(x_i) \prod_{k=i+1}^{r-1} R_0(x_k)>$ for $i = 1$ to $r-1$.

c) Elements obtained as linear combinations of the filters of above types.

3) By making use of the 1–D results, a procedure for the identification of 2–D cyclically closed abelian P–I filters is obtained.

4) It is shown that under CRT mapping cyclic and abelian group algebras are isomorphic and that any cyclic P–I filter whose dimension is a product of pair wise relatively prime integers has an equivalent abelian P–I filter realization. An important observation here is that there is an implementation advantage in using GWHT compared to G–T algorithm approach in realizing a cyclic P–I filter.

From the above results, it is concluded that in general GWHT can be used to extract only few cyclic features. These are the features of cyclically closed abelian P–I filters. However, if a GWHT matrix is a direct product of pairwise relatively prime length DFT matrices, then that GWHT can be used to extract all the corresponding cyclic features.

4.2 Suggestions for further research

1) In this thesis, we considered characterization of P–I filters that are invariant relative to a cyclic TAP group and a non–cyclic TAP group. Investigation can be done for

characterizing P–I filters that are invariant relative to a pair of non–cyclic TAP groups.

2) Based on the method of characterizing 1–D cyclically closed abelian P–I filters and experimental verifications, a procedure is given for identifying 2–D cyclically closed abelian P–I filters. Therefore, one can work out the theoretical details concerning the characterization of 2–D cyclically closed abelian P–I filters.

3) It is interesting to investigate whether there are any 2–D P–I filters that are invariant relative to two pairs of non–cyclic TAP groups.

4) It remains to be seen whether the number of zero crossings of the basis functions of GWHT can provide a necessary basis for developing the notions of filtering using P–I filters analogous to that of cyclic and dyadic P–I filters.

REFERENCES

1. Siddiqi. M. U. and V. P. Sinha., 'Signals and Systems over Finite Groups and Monoids', A Short Course on Digital Signal Processing for BEL Engineers, Dept. of Electrical Engineering, IIT Kanpur, 1988.
2. Henning F.Harmuth., 'Transmission of Information by Orthogonal Functions', Springer–Verlag, New York, 1969.
3. Siddiqi. M. U., 'A Study of Permutation–Invariant Linear Systems', Ph.D. Thesis, Dept. of Electrical Engineering, IIT Kanpur, 1976.
4. V. P. Sinha., 'Signal Spaces for Digital Filters', A Short Course on Digital Signal Processing for BEL Engineers, Dept. of Electrical Engineering, IIT Kanpur, 1988.
5. Huffman and Kunze., 'Linear Algebra', Prentice Hall of India, New Delhi, 1986.
6. Sundar Rajan. B., 'Spectral Characterization of Abelian group codes', M.Tech. Thesis, Dept. of Electrical Engineering, IIT Kanpur, 1984.
7. Madhusudhana. H. S., 'On Abelian codes which are closed under Cyclic shifts', M.Tech. Thesis, Dept. of Electrical Engineering, IIT Kanpur, 1987.
8. T. Beth., 'Generalizing the Discrete Fourier Transform', Discrete Math., vol.56, no. 2–3, pp 95–101, 1985.
9. P.R.K. Rao., 'Certain Generalizations of Permutation–Invariant Systems', Ph.D. Thesis, Dept. of Electrical Engineering, IIT Kanpur, 1978.